Usual topology on $\mathbb{R}$:

Let A be a subset of the set of real numbers $\mathbb{R}$. Then A is said to be open in $\mathbb{R}$ if and only if each $x \in A$ there exist an open interval I_x such that $x \in I_x \subseteq A$.

Let T be the class of all open set of real numbers. Then T is a topology on $\mathbb{R}$, called usual topology. Similarly we can define usual topology on $\mathbb{R}^2$.

Metric space:

Let X be a non empty set. A function $d : X \times X \longrightarrow \mathbb{R}$ is called a metric on X if it satisfies the following axioms

(i) $d(x, y) \geq 0$

(ii) $d(x, y) = 0 \Leftrightarrow x = y$

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y, z \in X$.

Then the pair (X, d) is called a metric space.

Open spheres in a metric space (X, d):

Let $x_0 \in X$ and $\epsilon > 0$. Then an open sphere centered at x_0 with radius ϵ is defined as,

$$S_\epsilon(x_0) = \{x \in X : d(x, x_0) < \epsilon\}$$

Open set in a metric space (X, d):

Let $G \subset X$. Then G is said to be open in X if $\forall x_0 \in X$, $\exists$ an open sphere $S_\epsilon(x_0)$, $\epsilon > 0$, such that

$$x_0 \in S_\epsilon(x_0) \subseteq G.$$

i.e $G \subseteq X$ is open in X if $G =$ union of open spheres

$$= \bigcup_{x \in G} S_2(x).$$

Metric topology:

Let d be a metric on a non empty set X. The topology T on X generated by the class of open spheres in X is called the metric topology on the topology induced by the metric d. Therefore, every metric space is a topological space.

Metrizable space:

A topological space (X, T) is said to be metrizable if there exist at least one metric d on X such that the class of its open set with respect to the metric d is precisely the given topology.

Intersection and union of topologies:

Let $\{T_\alpha\}$ be any collection of topologies on a non empty set X. Then $\bigcap T_\alpha$ is also a topology on X. But the union of any collection of topologies is not necessarily a topology.

e.g Let $X = \{a, b, c, d, e\}$

$T_1 = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, X\}$

$T_2 = \{\phi, \{a, b\}, \{c, d, e\}, X\}$

Then $T_1 \cup T_2 = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, \{c, d, e\}, X\}$ is not a topology on X.

Subspace topology or relative topology:

Let (X, T) be a topological space. If Y is a non empty subset of X, then the collection $T_Y = \{Y \cap G : G \in T\}$ is a topology on Y. called relative topology or subspace topology.

(i) Since $\phi \in T$ and $\phi = \phi \cap Y$

$\therefore$ $\phi \in T_Y$

Again, $Y \subseteq X$ and $X \in T$, therefore $Y \cap X = Y$

so $Y \in T_Y$

(ii) Let $\{Y \cap G_\alpha\}$ be a arbitrary family of ~~member~~ elements of T_Y.

$\therefore$ G_α are belongs to T, $\forall \alpha$

$\therefore$ $\bigcup_\alpha G_\alpha$ ~~belog~~ belongs to T, [$\because T$ is a topology on X]

$\Rightarrow$ $Y \cap (\bigcup_\alpha G_\alpha) \in T_Y$

$\Rightarrow$ $\bigcup_\alpha (Y \cap G_\alpha) \in T_Y$

(ii) Let $\{Y \cap G_1, Y \cap G_2, \ldots, Y \cap G_m\}$ be a finite sub collection of T_Y.

$\therefore$ $G_1, G_2, \ldots, G_m$ are belongs to T

$\therefore$ $\bigcap_{i=1}^{m} G_i \in T$

$\Rightarrow$ $Y \cap \left(\bigcap_{i=1}^{m} G_i\right) \in T_Y$

$\Rightarrow$ $\bigcap_{i=1}^{m} (Y \cap G_i) \in T_Y$

Example:

Let $X = \{a, b, c, d, e\}$

$T = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$

Then (X, T) is a topological space.

Let $Y = \{a, d, e\}$, then

$Y \cap \phi = \phi$, $Y \cap \{a\} = \{a\}$, $Y \cap \{c, d\} = \{d\}$, $Y \cap \{b, c, d, e\} = \{d, e\}$

$Y \cap X = Y$, $Y \cap \{a, c, d\} = \{a, d\}$

$\therefore T_Y = \{\phi, \{a\}, \{d\}, \{a, d\}, \{d, e\}, Y\}$

Then T_Y is the relative topology of Y.