2.9. Dissipation in energy:

We shall now calculate the energy which is Dissipated (wasted) in a viscous fluid in motion on account of internal friction. If $\vec{F}$ be the external force acting within a volume V of fluid and $t_i^{(\hat{n})}$ be the ith component of stress at any point on the bounding surface S of V, then rate of work done by $\vec{F}$ and $t_i^{(\hat{n})}$ is

$$\int_V \vec{q}\cdot\vec{F}\rho\, dV + \int_S t_i^{(\hat{n})} q_i\, ds \quad ;$$

$$W = \vec{F}\cdot d\vec{s}$$
$$\Rightarrow \frac{dW}{dt} = \dot{\vec{F}}\, d\vec{s} + \vec{F}\frac{d}{dt}(d\vec{s})$$
$$\Rightarrow \frac{dW}{dt} = \vec{F}\cdot\vec{q}.$$

Let W be the rate of work done per unit time by the external force $\vec{F}$ and the stress force $t_i^{(\hat{n})}$ then

$$W = \int_V \vec{q}\cdot\vec{F}\rho\, dV + \int_S t_i^{(\hat{n})} q_i\, ds$$

$$= \int_V \vec{q}\cdot\vec{F}\rho\, dV + \int_S (\sigma_{ki} n_k) q_i\, ds \quad ; \quad (\sigma_{ki} q_i)_{,k} = \frac{\partial}{\partial x_k}(\sigma_{ki} q_i)$$

$$= \int_V \vec{q}\cdot\vec{F}\rho\, dV + \int_V \frac{\partial}{\partial x_k}(\sigma_{ki} q_i)\, dV$$

$$= \int_V \vec{q}\cdot\vec{F}\rho\, dV + \int_V \frac{\partial\sigma_{ki}}{\partial x_k} q_i\, dV + \int_V \sigma_{ki}\frac{\partial q_i}{\partial x_k}\, dV$$

$$= \int_V \left[\vec{q}\cdot\vec{F}\rho + \frac{\partial\sigma_{ki}}{\partial x_k} q_i\right] dV + \int_V \sigma_{ki}\frac{\partial q_i}{\partial x_k}\, dV \longrightarrow (7)$$

Now, K.E. $= K = \frac{1}{2}\int_V \rho q^2\, dV$

$$\therefore \frac{DK}{Dt} = \int_V \frac{1}{2}\frac{D}{Dt}(q^2)\rho\, dV + \frac{1}{2}\int_V q^2 \frac{D}{Dt}(\rho\, dV)$$

$$= \frac{1}{2}\int_V \frac{D}{Dt}(q_i^2)\rho\, dv$$

$$= \int_V \rho \vec{q}_i \frac{D\vec{q}_i}{Dt} dv$$

$$= \int_V \left[\vec{q}_i \cdot \rho \vec{F} + q_i \frac{\partial \sigma_{ji}}{\partial x^j}\right] dv \qquad ; \left|\; \rho\frac{D\vec{q}_i}{Dt} = \rho\vec{F}_i + \frac{\partial \sigma_{ji}}{\partial x^j} \right.$$

equation of motion

Thus (1) can be written as

$$W = \frac{DK}{Dt} + \int_V \sigma_{ki} \frac{\partial q_i}{\partial x_k} dv \longrightarrow (ii)$$

This shows that the total work done by the external force and surface force donot contribute to increasing the kinetic energy of the fluid, some work is lost due to internal expansion and viscous forces. The 2nd part of R.S. of (ii) represent this loss.

Now, $\int_V \sigma_{ki} \frac{\partial q_i}{\partial x_k} dv = \int_V \sigma_{ji} \frac{\partial q_i}{\partial x_j} dv$

$$= \int_V \left[-p\delta_{ij} + 2\mu\left\{-\frac{1}{3}\delta_{ij} D_{kk} + D_{ij}\right\}\right] \frac{\partial q_i}{\partial x_j} dv$$

$$= -\int_V p\frac{\partial q_i}{\partial x_i} dv + \int_V \left[2\mu\left\{-\frac{1}{3}\delta_{ij} D_{kk}\frac{\partial q_i}{\partial x_j} + D_{ij}\frac{\partial q_i}{\partial x_j}\right\}\right] dv$$

$$\therefore \int_V \sigma_{ki}\frac{\partial q_i}{\partial x_k} dv = -\int_V p\theta\, dv - \int_V \frac{2\mu}{3}\theta^2 dv + 2\int_V \mu D_{ij}\frac{\partial q_i}{\partial x_j} dv \rightarrow (iii)$$

$$\left[\because D_{kk} = \operatorname{div}\vec{q} = \theta\right]$$

when there is no addition of heat, the thermodynamic equation gives,

$$dQ = 0 = dU + p\, d\tau .$$

when $\tau = \frac{1}{\rho}$, U = internal energy of gas.

$$\Rightarrow dU + p\,d\tau = 0$$

$$\Rightarrow dU = -p\,d\tau$$

$$\Rightarrow U = -\int^{\tau} p\,d\tau$$

$$\therefore \frac{DU}{Dt} = -p\frac{D\tau}{Dt} = -p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

$$= p\cdot\frac{1}{\rho^2}\frac{D\rho}{Dt}$$

$$= -\frac{p}{\rho}\theta.$$

[Equation of continuity $\frac{1}{\rho}\frac{D\rho}{Dt} + \operatorname{div}\vec{q} = 0$

$$\Rightarrow \frac{1}{\rho}\frac{D\rho}{Dt} + \theta = 0$$

$$\Rightarrow \frac{1}{\rho}\frac{D\rho}{Dt} = -\theta.\ ]$$

$\therefore$ Lost of energy $= \int_V \sigma_{ki}\frac{\partial q_i}{\partial x_k}\,dv$

$$= \int_V \frac{DU}{Dt}\rho\,dv - \int_V \frac{2\mu}{3}\theta^2\,dv + 2\int_V \mu\,D_{ij}\frac{\partial q_i}{\partial x_j}\,dv.$$

Last two terms actually means ~~of~~ loss ~~of~~ in energy due to viscous force. We ~~get~~ take μ to be constant and hence loss of energy due to viscosity is

$$= \mu\left[\int_V\left(-\frac{2}{3}\right)\theta^2\,dv + 2\int_V D_{ij}\frac{\partial q_i}{\partial x_j}\,dv\right]$$

$$= \mu\int_V\left[-\frac{2}{3}\theta^2 + 2\left\{D_{11}\frac{\partial u}{\partial x} + D_{22}\frac{\partial v}{\partial y} + D_{33}\frac{\partial w}{\partial z}\right\} + 2\left\{D_{12}\frac{\partial q_1}{\partial x_2} + D_{13}\frac{\partial q_1}{\partial x_3} + D_{21}\frac{\partial q_2}{\partial x_1} + D_{23}\frac{\partial q_2}{\partial x_3} + D_{31}\frac{\partial q_3}{\partial x_1} + D_{32}\frac{\partial q_3}{\partial x_2}\right\}\right]dv.$$

$$= \mu \int_V \Big[-\frac{2}{3}\theta^2 + 2\Big\{ D_{11}\frac{\partial u}{\partial x} + D_{22}\frac{\partial v}{\partial y} + D_{33}\frac{\partial w}{\partial z} \Big\}$$
$$+ 2\Big\{ D_{12}\frac{\partial u}{\partial y} + D_{13}\frac{\partial u}{\partial z} + D_{21}\frac{\partial v}{\partial x} + D_{23}\frac{\partial v}{\partial z} + D_{31}\frac{\partial w}{\partial x} + D_{32}\frac{\partial w}{\partial y} \Big\} \Big] dv$$

$$= \mu \int_V \Big[-\frac{2}{3}\theta^2 + 2\Big\{ D_{11}\frac{\partial u}{\partial x} + D_{22}\frac{\partial v}{\partial y} + D_{33}\frac{\partial w}{\partial z} \Big\} + 2D_{12}\Big\{ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \Big\}$$
$$+ 2D_{23}\Big\{ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \Big\} + 2D_{13}\Big\{ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \Big\} \Big] dv$$

$$= \mu \int_V \Big[-\frac{2}{3}\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \Big)^2 + 2\Big\{ \Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial y}\Big)^2 + \Big(\frac{\partial w}{\partial z}\Big)^2 + \Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \Big)^2$$
$$+ \Big(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \Big)^2 + \Big(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \Big)^2 \Big\} \Big] dv$$

$$= \mu \int_V \Big[\frac{4}{3}\Big\{ \Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial y}\Big)^2 + \Big(\frac{\partial w}{\partial z}\Big)^2 - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial w}{\partial z} - \frac{\partial v}{\partial y}\frac{\partial w}{\partial z} \Big\}$$
$$+ \Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \Big)^2 + \Big(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \Big)^2 + \Big(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \Big)^2 \Big] dv$$

$$= \mu \int_V \Big[\frac{2}{3}\Big\{ \Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \Big)^2 + \Big(\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} \Big)^2 + \Big(\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} \Big)^2 \Big\}$$
$$+ \Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \Big)^2 + \Big(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \Big)^2 + \Big(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \Big)^2 \Big] dv$$

Now, the integrand is +ve and it can not be zero. But, the integrand will vanish if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0, \quad \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0$$

i.e. when each component of D_{ij} is zero. In the case $D_{ij} = 0$, the motion is purely rotational without strain i.e. when there is solid body rotation. Besides the integrand may vanish if $u=v=w=$ constant, a mean translation.

There is another simplified form for this loss of energy. In case of Liquid, μ, ρ constant.

$$\text{Lost of energy} = \int_V \sigma_{ki} \frac{\partial q_i}{\partial x_k} dv$$

$$= \int_V \mu \Big\{ 2\Big[\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial y}\Big)^2 + \Big(\frac{\partial w}{\partial z}\Big)^2\Big] + \Big[\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big)^2$$

$$+ \Big(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\Big)^2 + \Big(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\Big)^2\Big]\Big\} dv.$$

$$= \mu \int_V \Big\{ 2\Big[\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial y}\Big)^2 + \Big(\frac{\partial w}{\partial z}\Big)^2\Big] + \Big(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\Big)^2$$

$$+ \Big(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\Big)^2 + \Big(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\Big)^2 + 4\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + 4\frac{\partial v}{\partial z}\frac{\partial w}{\partial y}$$

$$+ 4\frac{\partial u}{\partial z}\frac{\partial w}{\partial x}\Big\} dv.$$

$$= \mu \int_V \zeta^2 dv + 2\mu \int_V \Big[\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial y}\Big)^2 + \Big(\frac{\partial w}{\partial z}\Big)^2$$

$$+ 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + 2\frac{\partial v}{\partial z}\frac{\partial w}{\partial y} + 2\frac{\partial w}{\partial x}\frac{\partial u}{\partial z}\Big] dv$$

$$= \mu \int_V \zeta^2 dv + 2\mu \int_V \Big[\frac{1}{2}\nabla^2(q^2) - \text{div}(\vec{q} \times \vec{\zeta})\Big] dv$$

$$= \mu \int_V \zeta^2 dv + 2\mu \int_S \Big[\frac{1}{2}\nabla q^2 - (\vec{q} \times \vec{\zeta})\Big] \cdot \vec{n}\, ds.$$

Here S is the surface enclosing the volume V. If the motion be such that the velocity on S are zero, then $u, v, w = 0$, $\nabla q^2 = 0$ on S.

Thus surface integral vanishes and we have

$$\text{loss of energy} = \mu \int_V \zeta^2 dv. \quad ; \quad \mu \iiint (\xi^2 + \zeta^2 + \eta^2)\, dx\, dy\, dz.$$

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