2.10: Non-dimensional form of Navier-stoke's equation (ρ constant):

Navier-stoke's equation is in general a dimensional equation. Dimension of every term is $\frac{\rho U^2}{L}$, where U is some typical (representative) velocity, L is some typical length and ρ is constant density.

Let us take the equation of motion in cartesian coordinates along x-axis. It is

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u \longrightarrow (1)$$

Let us introduce

$$u'=\frac{u}{U},\quad v'=\frac{v}{U},\quad w'=\frac{w}{U}$$

$$x'=\frac{x}{L},\quad y'=\frac{y}{L},\quad z'=\frac{z}{L},\quad t'=\frac{U}{L}t,\quad p'=\frac{p}{\rho U^2}.$$

where $u', v', w', x', y', z', t', p'$ represents non-dimensional velocity component, time and pressure etc. in the equation (1), then we have,

$$\frac{\partial(Uu')}{\partial\left(\frac{L}{U}t'\right)}+(Uu')\frac{\partial(Uu')}{\partial(Lx')}+(Uv')\frac{\partial(Uu')}{\partial(Ly')}+(Uw')\frac{\partial(Uu')}{\partial(Lz')}$$

$$=-\frac{1}{\rho}\frac{\partial(\rho U^2p')}{\partial(Lx')}+\nu\left[\frac{\partial^2}{\partial(L^2x'^2)}+\frac{\partial^2}{\partial(L^2y'^2)}+\frac{\partial^2}{\partial(L^2z'^2)}\right](Uu')$$

$$\Rightarrow \frac{U^2}{L}\frac{\partial u'}{\partial t'}+\frac{U^2}{L}u'\frac{\partial u'}{\partial x'}+\frac{U^2}{L}v'\frac{\partial u'}{\partial y'}+\frac{U^2}{L}w'\frac{\partial u'}{\partial z'}=-\frac{U^2}{L}\frac{\partial p'}{\partial x'}$$

$$+\frac{\nu U}{L^2}\left(\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}\right)u'$$

$$\Rightarrow \frac{U^2}{L}\left(\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}+w'\frac{\partial u'}{\partial z'}\right)=-\frac{U^2}{L}\frac{\partial p'}{\partial x'}+\frac{\nu U}{L^2}\left(\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}\right)u'$$

$$\left(\because \frac{\partial u}{\partial t}\approx\frac{U^2}{L}\frac{\partial u'}{\partial t'}\right)$$

Dividing by $\frac{U^2}{L}$, we have,

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + w'\frac{\partial u'}{\partial z'} = -\frac{\partial p'}{\partial x'} + \frac{\nu}{LU}\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right)u'$$

$$\Rightarrow \frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + w'\frac{\partial u'}{\partial z'} = -\frac{\partial p'}{\partial x'} + \frac{1}{R}\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right)u'$$

where $R = \frac{LU}{\nu}$ is the Reynold number.

Similarly, the equation of motion along y and z axis are given by

$$\frac{\partial v'}{\partial t'} + u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} + w'\frac{\partial v'}{\partial z'} = -\frac{\partial p'}{\partial y'} + \frac{1}{R}\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right)v'$$

and $$\frac{\partial w'}{\partial t} + u'\frac{\partial w'}{\partial x'} + v'\frac{\partial w'}{\partial y'} + w'\frac{\partial w'}{\partial z'} = -\frac{\partial p'}{\partial z'} + \frac{1}{R}\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right)w'$$

and the equation continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{(in non-dimensional form)}$$

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'} = 0$$

Navier-Stoke's equation is highly non-linear. It can be solved -

(i) exactly it non-linear terms cancelled out

(ii) when the non-linear term neglected retained partially with linear terms

(iii) When the equations are approximated to convenient forms. (Boundary layer approximation).

Besides, there are few well-known problems which are exactly solved with non-linear terms.

(i) Two dimensional steady flow between non-parallel plates (Jaffery - Hamel problem)

(II) Motion due to rotating disk [Karman's problem]

(III) Stagnatic flow [Hiemenz problem]

(IV) Flow from a circular hole [Lasdoll squire problem]

2.11:

The Reynold number:

We consider the Navier Stoke's equation is

$$\frac{\partial \vec{q}}{\partial t} + \frac{1}{2}\nabla q^2 - \vec{q}\times\vec{\xi} = \vec{F} - \frac{1}{\rho}\nabla p + \nu\nabla^2 q^2 \longrightarrow (1)$$

If the external force $\vec{F}$ is conservative, $\vec{F} = -\nabla\Omega$

and (1) can be written as

$$\frac{\partial \vec{q}}{\partial t} + \frac{1}{2}\nabla q^2 - \vec{q}\times\vec{\xi} = -\nabla\Omega - \frac{1}{\rho}\nabla p + \nu\nabla^2\vec{q}$$

$$= -\nabla\left(\Omega + \frac{p}{\rho}\right) + \nu\nabla^2\vec{q} \longrightarrow (II)$$

If we take the external force to produce hydrostatic pressure p_0 then for fluid at rest, we find

$$-\nabla\left(\Omega + \frac{p_0}{\rho}\right) = 0 \quad \text{i.e. } p_0 = -\rho\Omega$$

If we consider the pressure p as the sum of hydrostatic pressure p_0 and hydrodynamics pressure p_1 i.e. $p = p_0 + p_1$, $p_0 = -\rho\Omega$, then equation (II) becomes,

$$\frac{\partial \vec{q}}{\partial t} + \frac{1}{2}\nabla q^2 - \vec{q}\times\vec{\xi} = -\frac{1}{\rho}\nabla p_1 + \nu\nabla^2\vec{q}$$

Multiplying by ρ

$$\rho\left(\frac{\partial \vec{q}}{\partial t} + \frac{1}{2}\nabla q^2 - \vec{q}\times\vec{\xi}\right) = -\nabla p_1 + \mu\nabla^2\vec{q} \quad ; \because \mu = \rho\nu$$

This equation consists of three types of forces

(a) effective force $\rho\left\{\frac{\partial \vec{q}}{\partial t}+\frac{1}{2}\nabla q^2-\vec{q}\times\vec{\xi}\right\}$

(b) the pressure force $-\nabla p,$

(c) the viscous force $\mu\nabla^2\vec{q}$

i.e. Effective force = pressure force + viscous force.

Now, the dimension of the terms of

(a) is $\rho\frac{U^2}{L}$, U = typical velocity
L = " length
ρ = " dencity.

and the dimension of

$$\mu\nabla^2\vec{q}=\mu\frac{U}{L^2};$$

$\because \rho\frac{\partial\vec{q}}{\partial t}=\frac{\rho U^2}{L^2}$

$\rho\frac{1}{2}\nabla^2 q = \dots$

$\rho\vec{q}\times\vec{\xi} = \dots$

Therefore the ratio of effective force and viscous force is

$$\frac{\rho\frac{U^2}{L}}{\mu\frac{U}{L^2}}=\frac{UL}{\nu},$$ which represents a number and is called Reynold number.

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Two forces are similar when R is equal for both this principle was first enumerated by Reynold is connection with is investigation into the flow through pipes and is known as Reynold's principle of similarity.