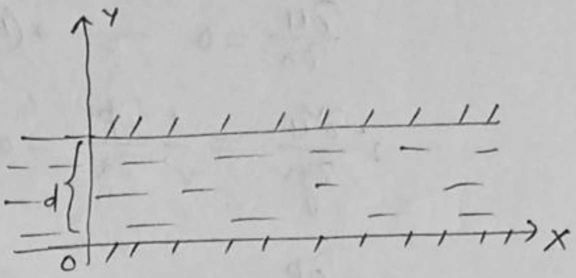
2.12: Steady flow in between parallel plates:



We consider liquid forced under pressure to move between two fixed parallel plates at a distance 'd' apart as shown in figure.

We take x-axis in the direction of the flow and y-axis perpendicular to its with origin at the lower plate.

Since the motion is in the x-direction only so that if $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$, then $v = 0$, $w = 0$.

Now, the general equation of continuity for liquid ρ = constant is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} = 0$$

Hence it is

$$\frac{\partial u}{\partial x} = 0$$

i.e. u is independent of x

$\therefore$ u is a function of y only.

The general equation of motion are (for steady case)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right]$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right]$$

In this case, we have,

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} = 0$$

$$-\frac{\partial p}{\partial y} = 0 \quad \text{and} \quad -\frac{\partial p}{\partial z} = 0 \quad .$$

i.e. the equation of continuity and motion are

$$\frac{\partial u}{\partial x} = 0 \longrightarrow (\mathrm{i})$$

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \longrightarrow (\mathrm{ii})$$

$$\frac{\partial p}{\partial y} = 0 \longrightarrow (\mathrm{iii})$$

$$\frac{\partial p}{\partial z} = 0 \longrightarrow (\mathrm{iv})$$

From (iii) and (iv), we have

$$p = p(x) \text{ only}$$

and from (i) we have

$$u = u(y) \text{ only}$$

So, We have to determine u and p from (ii).

Now, in (ii), the first term is a function of y only and the 2nd term is a function of x only. The equation is compatible for solution if

$$\nu \frac{\partial^2 u}{\partial y^2} = \text{constant} = k \text{ (say)}$$

From (ii) $\quad k - \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$

$$\Rightarrow \frac{\partial p}{\partial x} = k\rho$$

$$\Rightarrow p = k\rho x + \text{constant}.$$

$\therefore$ pressure varies as x varies.

Again, $\nu \frac{\partial^2 u}{\partial y^2} = k \Rightarrow \frac{d^2 u}{dy^2} = \frac{k}{\nu}$

solution is $\quad u = A + By + \frac{k}{2\nu}y^2, \longrightarrow (\mathrm{v})$

where A and B are constant.

Boundary conditions are

$u = 0$ at $y = 0$

$u = 0$ at $y = d$.

at $y = 0, u = 0$

from (v) we have $\quad 0 = A + 0 + 0$

$$\Rightarrow A = 0$$

at $y = d$, $u = 0$

$\therefore$ from (v) we have,

$$0 = A + Bd + \frac{k}{2\nu} d^2$$

$$\Rightarrow 0 = 0 + Bd + \frac{k}{2\nu} d^2$$

$$\Rightarrow B = -\frac{kd}{2\nu}.$$

Substituting in (v)

$$u = -\frac{kd}{2\nu} y + \frac{k}{2\nu} y^2$$

$$= \frac{k}{2\nu}\left[y^2 - yd\right]$$

$$= \frac{k}{2\nu}\left[\left(y - \frac{d}{2}\right)^2 - \frac{d^2}{4}\right]$$

$$= -\frac{k}{2\nu}\left[\frac{d^2}{4} - \left(y - \frac{d}{2}\right)^2\right]$$

Here k is negative and maximum value of

$$u = -k\frac{d^2}{8\nu}$$

If m be the total flow of mass across any section then,

$$m = \int_{y=0}^{d} \rho u\, dy = \rho \int_0^d \frac{k}{2\nu}\left(y^2 - yd\right) dy$$

$$= \frac{k\rho}{2\nu}\left[\frac{y^3}{3} - \frac{y^2 d}{2}\right]_0^d$$

$$= \frac{\rho K}{2\nu}\left[\frac{d^3}{3} - \frac{d^3}{2}\right]$$

$$= -\frac{\rho K}{2\nu}\cdot\frac{d^3}{6}$$

$$\Rightarrow m = -\frac{\rho K d^3}{12\nu}.$$

$$\Rightarrow K = -\frac{12\nu m}{\rho d^3}$$

Now, $u = \frac{K}{2\nu}(y^2 - yd)$

$$= -\frac{6m}{\rho d^3}(y^2 - yd)$$

$$= -\frac{6m}{\rho d^3}(y^2 - yd)$$

$\therefore$ viscous force is

$$\mu\frac{\partial u}{\partial y} = -\frac{6m\mu}{\rho d^3}[2y - d]$$

$\therefore$ The viscous force at $y = 0$ is $\frac{6m\mu d}{\rho d^3}$

and " " " " $y = d$ is $-\frac{6m\mu d}{\rho d^3}$

i.e. the viscous force at $y = 0$ is $= \frac{6m\mu}{\rho d^2}$ and

the viscous " " $y = d$ is $= -\frac{6m\mu}{\rho d^2}$

Again, $p = K\rho x + \text{constant}$

$$= -\frac{12\nu m x}{d^3} + \text{constant}$$

If the bed be inclined with the angle θ to the horizon, then equation for u along the bed is

$$\mu \frac{\partial^2 u}{\partial y^2} + \left(g\rho \sin\theta - \frac{\partial p}{\partial x}\right) = 0$$

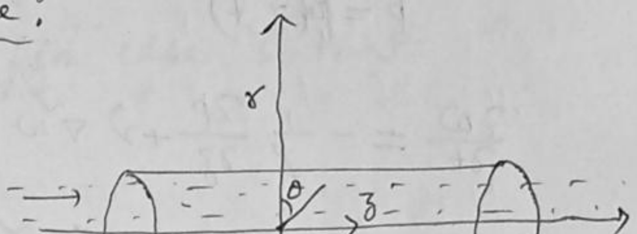
In this case,

$$g\rho \sin\theta - \frac{\partial p}{\partial x} = -\rho k \quad \text{(constant)}$$

and u is same (can be determined) as before.

#

2.13 Flow through a circular pipe:



Let us consider viscous liquid through a circular pipe whose axis is in the direction of z-axis and r-axis is taken perpendicular to it as shown in figure. Let u, v, w be the velocity components along r, θ and z axes respectively, then

$$u = v = 0, \quad w \neq 0.$$

Now, consider the equation of continuity

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{r\partial\theta} + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \frac{\partial w}{\partial z} = 0 \longrightarrow (1)$$

and equation of motion

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\nabla^2 u - \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{v}{r^2}\right]$$

$$\Rightarrow \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{\rho}\frac{\partial p}{r\partial\theta} + \nu\left[\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2}\right]$$