If the bed be inclined with the angle θ to the horizon, then equation for u along the bed is

$$\mu \frac{\partial^2 u}{\partial y^2} + \left(g\rho \sin\theta - \frac{\partial p}{\partial x} \right) = 0$$

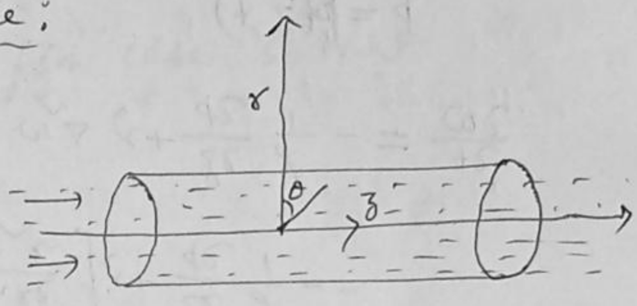
In this case,

$$g\rho \sin\theta - \frac{\partial p}{\partial x} = -\rho k \quad \text{(constant)}$$

and u is same (can be determined) as before.

#

2.13 Flow through a circular pipe:



Let us consider viscous liquid through a circular pipe whose axis is in the direction of z-axis and r-axis is taken perpendicular to it as shown in figure. Let u, v, w be the velocity components along r, θ and z axes respectively, then

$$u = v = 0 \ , \ w \neq 0 .$$

Now, consider the equation of continuity

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{r\partial \theta} + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \frac{\partial w}{\partial z} = 0 \longrightarrow (1)$$

and equation of motion

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\nabla^2 u - \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{v}{r^2}\right]$$

$$\Rightarrow \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{\rho}\frac{\partial p}{r\partial \theta} + \nu\left[\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2}\right]$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial \theta}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left[\nabla^2 w\right],$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}$

ie. $-\frac{1}{\rho}\frac{\partial p}{\partial r}=0 \longrightarrow$ (ii)

$-\frac{1}{\rho}\frac{\partial p}{r\partial \theta}=0 \longrightarrow$ (iii)

$\frac{\partial w}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w \longrightarrow$ (iv)

From the above equation, we have,

$w=w(r,t)$

$p=p(z,t)$

$$\frac{\partial w}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w$$

$$=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}\right]w$$

$$=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right] \longrightarrow (V)$$

This equation contains two unknowns w, p.

Now, we multiply both sides of $2\pi\rho r dr$ and integrating from $r=0$ to $r=a$, where a is the radius of the pipe.

$$\int_0^a \frac{\partial w}{\partial t}\cdot 2\pi\rho r dr=-\int_0^a -\frac{1}{\rho}\frac{\partial p}{\partial z}\cdot 2\pi\rho r dr+\int_0^a \nu\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right]\cdot 2\pi\rho r dr.$$

$$\Rightarrow \frac{\partial}{\partial t}\int_0^a w\, 2\pi\rho r dr=-\int_0^a \frac{\partial p}{\partial z}\, 2\pi r dr+\int_0^a \frac{\mu}{\rho}\frac{1}{r}\left[\frac{\partial}{\partial r}\left\{r\frac{\partial w}{\partial r}\right\}\right]\cdot 2\pi\rho r dr.$$

$$\Rightarrow \frac{\partial m}{\partial t} = -\frac{\partial p}{\partial z}\int_0^a 2\pi r\,dr + \mu . 2\pi\left[r\frac{\partial \omega}{\partial r}\right]_0^a, \quad \text{where } m = \int_0^a \omega . 2\pi\rho\, r\,dr$$

$$= -\frac{\partial p}{\partial z}\left[\pi r^2\right]_0^a + 2\pi\mu a\left(\frac{\partial \omega}{\partial r}\right)_{r=a}$$

$$\Rightarrow \frac{\partial m}{\partial t} = -\pi a^2\frac{\partial p}{\partial z} + 2\pi a\mu\left[\frac{\partial \omega}{\partial r}\right]_{r=a}$$

$\Rightarrow$ Rate of change of flow of total mass = - pressure gradient on the cross-section + Viscous stress on the boundary.

Case (I):

Let us consider the steady flow case. So that

$p = p(z)$, $\omega = \omega(r)$

then the equation (V) reduces to

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega}{\partial r}\right)\right] = 0 \longrightarrow (\text{VI})$$

Now, the 1st part of equation (VI) depends on z only and the 2nd part depends on r only. Therefore the equation is compatible for solution if

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = A.$$

From (VI), we have,

$$-A + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega}{\partial r}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial \omega}{\partial r}\right) = \frac{Ar}{\nu}$$

$$\Rightarrow \frac{d}{dr}\left(r\frac{d\omega}{dr}\right) = \frac{Ar}{\nu}$$

$$-\frac{dp}{dz} = P$$

the -ve sign is taken as convention.

$$\frac{d}{dr}\left(r\frac{\partial q}{\partial r}\right) = -\frac{Pr}{\mu}$$

Here $\vec{\omega} = \vec{q}$.

$$\therefore \; r\frac{dw}{dr} = \frac{Ar^2}{2\nu} + B \text{ (say)}$$

$$\Rightarrow \frac{dW}{dr} = \frac{A}{\nu}\cdot\frac{r}{2} + \frac{B}{r}$$

$$\Rightarrow W = \frac{A}{2\nu}\cdot\frac{r^2}{2} + B\log r + c$$

$$\Rightarrow W = \frac{A}{4\nu}\cdot r^2 + B\log r + c, \text{ where } A, B, C \text{ are constants}$$

Boundary conditions are

(a) $w = 0$ at $r = a$

and (b) w is continuous when $0 < r \leq a$.

From condition (b) we have $B = 0$, otherwise discontinuity will come along the axis of the pipe.

Thus from condition (a)

$$0 = \frac{A}{2\nu}\cdot\frac{a^2}{2} + c$$

$$\Rightarrow c = -\frac{Aa^2}{4\nu}$$

Hence, $w = \frac{A}{4\nu}(r^2 - a^2)$

Now, we are to determine A. If pressure gradient $\frac{\partial p}{\partial z}$ is known then A is known. Otherwise A can be found if the total flow of mass across any section is known.

Now, total flow of mass across a section of the pipe is

$$m = \int_0^a 2\pi r w \rho \, dr = \int_0^a \frac{2\pi\rho A}{4\nu}(r^2 - a^2)\, r\, dr$$

$$= \frac{2\pi\rho A}{4\nu}\left[\frac{r^4}{4} - a^2\frac{r^2}{2}\right]_0^a$$

$$= \frac{2\pi \rho A}{4\nu}\left[\frac{a^4}{4} - \frac{a^4}{2}\right]$$

$$= \frac{\pi \rho A}{2\nu}\left(-\frac{a^4}{4}\right)$$

$$= -\frac{A\pi \rho a^4}{8\nu}$$

Since m is positive, therefore A is $-$ve (i.e. pressure gradient is $-$ve) and hence

$$A = -\frac{8\nu m}{\pi \rho a^4}$$

$$\therefore W = -\frac{8\nu m}{\pi \rho a^4} \cdot \frac{1}{4\nu}(r^2 - a^2)$$

$$= \frac{2m}{\pi \rho a^4}(a^2 - r^2).$$

This is a parabolic velocity profile.

At $r = a$, $W = 0$

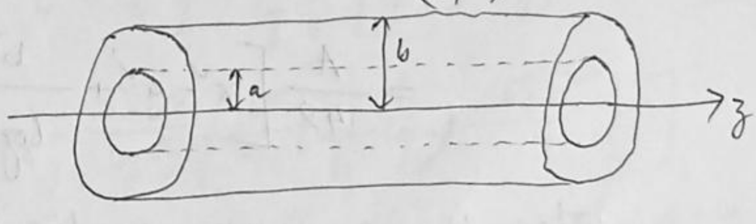
and at $r = 0$, $W = \frac{2m}{\pi \rho a^2}$

i.e. W is maximum.

[This experiment was done by Hagen in 1839 and afterwards by Poiseuille in 1840 and the flow is known as Hagen-Poiseuille flow.]

Case (II): Steady flow between coaxial circular pipes:

$(b > a)$



We get the solution of W as

$$W = \frac{A}{4\nu} r^2 + B \log r + C.$$

Let the flow takes places between two coaxial circular pipes of radii a and b, $b>a$ as shown in figure. The boundary conditions on W are

$$W = 0 \quad \text{when } r = a$$
$$W = 0 \quad \text{when } r = b.$$

when $r = a$, we have,

$$0 = \frac{A}{4\nu} a^2 + B \log a + C$$

When $r = b$, we have,

$$0 = \frac{A}{4\nu} b^2 + B \log b + C$$

subtracting we get,

$$0 = \frac{A}{4\nu}(a^2 - b^2) + B\left[\log a - \log b\right]$$

$$\Rightarrow B \log\frac{a}{b} = \frac{A}{4\nu}(b^2 - a^2)$$

$$\Rightarrow B = \frac{\frac{A}{4\nu}(b^2 - a^2)}{\log\frac{a}{b}}$$

and
$$C = -\frac{Aa^2}{4\nu} - \frac{A}{4\nu}\,\frac{b^2 - a^2}{\log\frac{a}{b}}\cdot \log a$$

Therefore

$$W = \frac{A}{4\nu} r^2 + \frac{A}{4\nu}\,\frac{(b^2 - a^2)}{\log\frac{a}{b}} \log r - \frac{Aa^2}{4\nu} - \frac{A}{4\nu}\cdot\frac{b^2 - a^2}{\log\frac{a}{b}}\cdot \log a$$

$$= \frac{A}{4\nu}\left[r^2 - a^2 + \frac{b^2 - a^2}{\log\frac{a}{b}} \log\frac{r}{a}\right]$$

when the pressure gradient is known, then A is known, otherwise A is to be determined from the value of total flow of mass across any section of the pipe.

$$\therefore \quad m = \int_{r=a}^{b} w\, 2\pi\rho\, r\, dr$$

$$= 2\pi\rho \int_{r=a}^{b} \frac{A}{4\nu}\left[r^2 - a^2 + \frac{b^2-a^2}{\log\frac{a}{b}} \log\frac{r}{a}\right] r\, dr$$

$$= 2\pi\rho\, \frac{A}{4\nu}\left\{ \left[\frac{r^4}{4} - a^2\frac{r^2}{2}\right]_{r=a}^{b} + \frac{b^2-a^2}{\log\frac{a}{b}} \int_{r=a}^{b} \log\frac{r}{a}\, r\, dr \right\}$$

Now, $\displaystyle\int_{r=a}^{b} \left[\log r - \log a\right] r\, dr$

$$= \int_{r=a}^{b} r\log r\, dr - \int_{r=a}^{b} \log a \cdot r\, dr$$

$$= \log r \cdot \frac{r^2}{2}\Big]_a^b - \int_a^b \frac{1}{r}\,\frac{r^2}{2}\, dr - \log a\left[\frac{r^2}{2}\right]_a^b$$

$$= \frac{b^2}{2}\log b - \frac{a^2}{2}\log a - \frac{1}{2}\left[\frac{r^2}{2}\right]_a^b - \log a\left[\frac{b^2-a^2}{2}\right]$$

$$= \frac{b^2}{2}\log b - \frac{a^2}{2}\log a - \frac{1}{4}(b^2-a^2) - \frac{b^2}{2}\log a + \frac{a^2}{2}\log a$$

$$= -\frac{1}{4}(b^2-a^2) + \frac{b^2}{2}(\log b - \log a)$$

$$\therefore \quad m = 2\pi\rho\, \frac{A}{4\nu}\left\{ \frac{b^4}{4} - \frac{a^2b^2}{2} - \frac{a^4}{4} + \frac{a^4}{2}\right] + \frac{b^2-a^2}{\log\frac{a}{b}}\left[\frac{b^2}{2}(\log b - \log a) - \frac{1}{4}(b^2-a^2)\right]$$

$$= \frac{A\pi\rho}{2\nu}\left[\frac{b^4-a^4}{4} - \frac{a^2}{2}(b^2-a^2) + \frac{b^2-a^2}{\log\frac{a}{b}}\left\{\frac{b^2}{2}(\log b - \log a) - \frac{b^2-a^2}{4}\right\}\right]$$

Thus A may be obtained in terms of m and substituting A we may obtain W also.