Construction of a uniform vector field:

Let A_i be a covariant vector. Then its x^j- covariant derivative is given by

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - \Gamma_{ij}^{\alpha} A_\alpha$$

Let $A_{i,j} = 0$. Then we have

$$\frac{\partial A_i}{\partial x^j} = \Gamma_{ij}^{\alpha} A_\alpha \longrightarrow (1)$$

$$\Rightarrow \frac{\partial A_i}{\partial x^j} dx^j = \Gamma_{ij}^{\alpha} A_\alpha dx^j$$

$$\Rightarrow dA_i = \Gamma_{ij}^{\alpha} A_\alpha dx^j \longrightarrow (2)$$

$$\Rightarrow A_i = \int dA_i = \int \Gamma_{ij}^{\alpha} A_\alpha dx^j$$

This shows that the equation (1) is integrable only when the R.H.S of (2) is a perfect differential.

So, let $\Gamma_{ij}^{\alpha} A_\alpha dx^j = dB_i$

Then $\Gamma_{ij}^{\alpha} A_\alpha dx^j = \frac{\partial B_i}{\partial x^j} dx^j$

$$\Rightarrow \left(\Gamma_{ij}^{\alpha} A_\alpha - \frac{\partial B_i}{\partial x^j}\right) dx^j = 0$$

Since x^j is arbitrary, therefore we must have

$$\Gamma_{ij}^{\alpha} A_\alpha = \frac{\partial B_i}{\partial x^j}$$

Differentiating both sides w.r.to x^k, we get

$$\frac{\partial \Gamma_{ij}^{\alpha}}{\partial x^k} A_\alpha + \Gamma_{ij}^{\alpha} \frac{\partial A_\alpha}{\partial x^k} = \frac{\partial^2 B_i}{\partial x^k \partial x^j} \longrightarrow (3)$$

Interchanging the indices j and k, we get

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$$\frac{\partial \Gamma_{ik}^{\alpha}}{\partial x^j} A_\alpha + \Gamma_{ik}^{\alpha} \frac{\partial A_\alpha}{\partial x^j} = \frac{\partial^2 B_i}{\partial x^j \partial x^k} \longrightarrow (4)'$$

$$(4)' - (3) \Rightarrow -\frac{\partial \Gamma_{ij}^{\alpha}}{\partial x^k} A_\alpha + \frac{\partial \Gamma_{ik}^{\alpha}}{\partial x^j} A_\alpha - \Gamma_{ij}^{\alpha} \frac{\partial A_\alpha}{\partial x^k} + \Gamma_{ik}^{\alpha} \frac{\partial A_\alpha}{\partial x^j} = 0$$

Using (1), we get

$$-\frac{\partial \Gamma_{ij}^{\alpha}}{\partial x^k} A_\alpha + \frac{\partial \Gamma_{ik}^{\alpha}}{\partial x^j} A_\alpha - \Gamma_{ij}^{\alpha} (\Gamma_{\alpha k}^{\beta} A_\beta) + \Gamma_{ik}^{\alpha} (\Gamma_{\alpha j}^{\beta} A_\beta) = 0$$

$$\Rightarrow -\frac{\partial \Gamma_{ij}^{\alpha}}{\partial x^k} A_\alpha + \frac{\partial \Gamma_{ik}^{\alpha}}{\partial x^j} A_\alpha - \Gamma_{ij}^{\beta} \Gamma_{\beta k}^{\alpha} A_\alpha + \Gamma_{ik}^{\beta} \Gamma_{\beta j}^{\alpha} A_\alpha = 0$$

$$\Rightarrow A_\alpha \left(-\frac{\partial \Gamma_{ij}^{\alpha}}{\partial x^k} + \frac{\partial \Gamma_{ik}^{\alpha}}{\partial x^j} - \Gamma_{ij}^{\beta} \Gamma_{\beta k}^{\alpha} + \Gamma_{ik}^{\beta} \Gamma_{\beta j}^{\alpha} \right) = 0$$

$$\Rightarrow A_\alpha R_{ijk}^{\alpha} = 0 \longrightarrow (5)$$

where R_{ijk}^{α} is the Riemann-Christoffel curvature tensor.

Since A_α being arbitrary, therefore from (5), we get

$$R_{ijk}^{\alpha} = 0.$$

Thus when $R_{ijk}^{\alpha} = 0$, then from (2) we see that $\int dA_{i}$ between any two points is independent of the path of transfer. Then we can carry the vector A_i by parallel displacement to any point obtaining a unique result independent of the path of transfer. When a vector is displaced in this way, we obtain a uniform vector field.

This shows that the construction of a uniform vector field is only possible when the Riemann-christoffel curvature tensor vanishes.

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Note: When the Riemann-christoffel curvature tensor vanishes, Then the differential equations $A_{i,j}=0$ and $\frac{\partial A_i}{\partial x^j} - \Gamma^{\alpha}_{ij} A_\alpha = 0$ are integrable.

condition for flat space-time:

Statement:

The vanishing of the Riemann-christoffel curvature tensor is a necessary and sufficient condition for a space-time to be flat.

proof: Let a space-time be flat. Then a coordinate system can be found in it for which the fundamental tensors g_{ij} are constants.

Now when g_{ij} are constants, the christoffel symbols all vanish. But the christoffel symbols are not tensors, therefore these will not in general continue to vanish when other coordinates are substituted in the ~~same~~ flat space-time.

Again when g_{ij} are constants, the Riemann-christoffel curvature tensor, being composed of derivatives and products of christoffel symbols, continue to vanish when other coordinates are substituted in the same flat space time. Thus the vanishing of Riemann-christoffel curvature tensor is a necessary condition for a space-time to be flat.