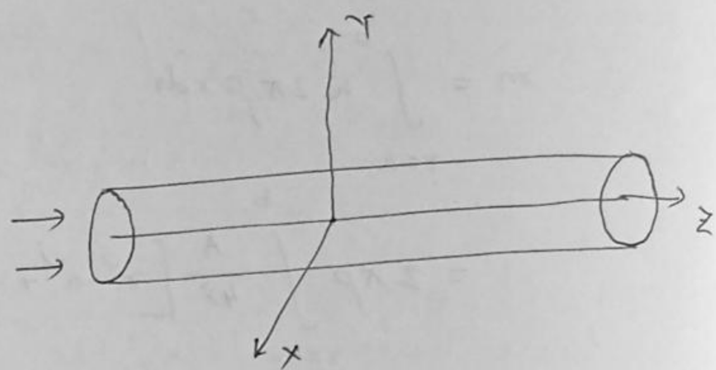
Case (III):

Steady flow in pipes of elliptic cross-section:



Let us consider the incompressible viscous fluid through a pipe having elliptic cross-section.

Also let z-axis be taken in the direction of flow along the axis of the pipe. In steady case, the equation of continuity and motion are.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v$$

& $$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

Here, $u=v=0$, $w\neq 0$.

therefore the above equation reduce to

$$\frac{\partial w}{\partial z}=0 \longrightarrow (\mathrm{i})$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=0 \longrightarrow (\mathrm{ii})$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y}=0 \longrightarrow (\mathrm{iii})$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right]=0 \longrightarrow (\mathrm{iv})$$

from (ii) & (iii), p is independent of x and y only,

therefore $p=p(z)$.

Now, the equation (iv) is solvable if each term is constant

i.e. $-\frac{1}{\rho}\frac{\partial p}{\partial z} = A$ (constant)

$$\therefore \quad \nu\left[\frac{\partial^2 \omega}{\partial x^2}+\frac{\partial^2 \omega}{\partial y^2}\right]+A=0$$

$$\Rightarrow \frac{\partial^2 \omega}{\partial x^2}+\frac{\partial^2 \omega}{\partial y^2} = -\frac{A}{\nu} \longrightarrow \text{(vi)}$$

This equation of w is to be solved under the boundary conditions

$\omega = 0$ at $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

and ω is continuous in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

We take the trial solution of w as

$$W = B\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right) \longrightarrow \text{(vii)}$$

This satisfies the boundary conditions and we fit it in the differential equation to find the constant B.

Putting this value of W in (vi), we get,

$$B\left(-\frac{2}{a^2}\right)+B\left(-\frac{2}{b^2}\right) = -\frac{A}{\nu}$$

$$\Rightarrow 2B \cdot \frac{a^2+b^2}{a^2 \times b^2} = \frac{A}{\nu}$$

$$\Rightarrow B = \frac{A}{2\nu}\left(\frac{a^2b^2}{a^2+b^2}\right)$$

Substituting in (vii),

$$W = \frac{A}{2\nu}\left(\frac{a^2b^2}{a^2+b^2}\right)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right).$$

Now, the flux of the fluid over the area of the ellipse is given by

$$\iint W\,dxdy$$

$$= \iint \frac{A}{2\nu} \frac{a^2b^2}{a^2+b^2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dxdy,$$

where y varies from $-\frac{b}{a}\sqrt{a^2-x^2}$ to $\frac{b}{a}\sqrt{a^2-x^2}$ and

x varies from $-a$ to a.

Now,

$$\iint W\,dxdy$$

$$= \frac{A}{2\nu}\frac{a^2b^2}{a^2+b^2}\left[\iint dxdy - \frac{1}{a^2}\iint x^2\,dxdy - \frac{1}{b^2}\iint y^2\,dxdy\right]$$

Now,

$$\int_{x=-a}^{a}\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} dxdy = \int_{-a}^{a}\Big[y\Big]_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \int_{-a}^{a}\frac{2b}{a}\sqrt{a^2-x^2}\,dx$$

$$= \frac{2b}{a}\int_{\pi}^{0}\sqrt{a^2-a^2\cos^2\theta}\,(-a\sin\theta)\,d\theta$$

Putting $x = a\cos\theta$
$dx = -a\sin\theta\,d\theta$.

$$= 2ab\,\frac{1}{2}\int_{0}^{\pi}(1-\cos 2\theta)\,d\theta$$

$$= ab\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= ab\left[\pi - 0\right]$$

$$= \pi ab.$$

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$$\iint x^2 dxdy = \int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} x^2 dxdy$$

$$= \int_{-a}^{a} x^2 \left[y\right]_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \int_{-a}^{a} \frac{2b}{a} x^2\sqrt{a^2-x^2}\, dx$$

$$= \int_{\pi}^{0} \frac{2b}{a} \cdot a^2\cos^2\theta . a\sin\theta \,(-a\sin\theta)\, d\theta .$$

$$= 2a^3 b \int_0^{\pi} \cos^2\theta \sin^2\theta \, d\theta$$

$$= 2a^3b . 2 \int_0^{\frac{\pi}{2}} \cos^2\theta \sin^2\theta \, d\theta .$$

$$= 4a^3b . \frac{\pi}{16}$$

$$= \frac{\pi a^3 b}{4}$$

and $$\iint y^2 dxdy = \int_{-a}^{a} \left[\frac{y^3}{3}\right]_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{3}\int_{-a}^{a} \left[\frac{b^3}{a^3}\left(\sqrt{a^2-x^2}\right)^3 + \frac{b^3}{a^3}\left(\sqrt{a^2-x^2}\right)^3\right] dx$$

$$= \frac{2b^3}{3a^3}\int_{-a}^{a} \left(\sqrt{a^2-x^2}\right)^3 dx$$

$$= \frac{2b^3}{3a^3}\int_{\pi}^{0} a^3\sin^3\theta . (-a\sin\theta\, d\theta)$$

$$= \frac{2b^3a}{3}\int_0^{\pi} \sin^4\theta\, d\theta . = \frac{2ab^3}{3} . 2\int_0^{\pi/2} \sin^4\theta\, d\theta = \frac{2ab^3}{3} . 2 . \frac{3}{4} . \frac{1}{2} \frac{\pi}{2}$$

$$= \frac{\pi a b^3}{4}$$

Hence,

$$\int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} W\,dx\,dy = \frac{A}{2\nu}\,\frac{a^2b^2}{a^2+b^2}\left[\pi ab - \frac{1}{a^2}\,\frac{\pi a^3 b}{4} - \frac{1}{b^2}\cdot\frac{\pi a b^3}{4}\right]$$

$$= \frac{A}{2\nu}\,\frac{a^2b^2}{a^2+b^2}\left[\pi ab - \frac{\pi ab}{4} - \frac{\pi ab}{4}\right]$$

$$= \frac{A\pi}{4\nu}\,\frac{a^3b^3}{a^2+b^2}.$$

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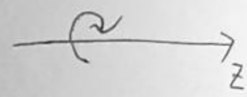
Steady rotation about the axis:

Let us consider the steady rotation of the incompressible viscous fluid about the axis of z. Suppose u, v, w be the velocity components along r, θ, z increasing direction. we have considered rotation only therefore



$$u = w = 0, \quad v \neq 0.$$

Now, the equation of continuity and motion in cylindrical polar coordinates (r, θ, z) are

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{r\partial\theta} + \frac{\partial w}{\partial z} = 0$$

$$u\frac{\partial u}{\partial r} + v\frac{\partial u}{r\partial\theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\nabla^2 u - \frac{2}{r^2}\frac{\partial v}{\partial\theta} - \frac{u}{r^2}\right]$$

$$u\frac{\partial v}{\partial r} + v\frac{\partial v}{r\partial\theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{\rho}\frac{\partial p}{r\partial\theta} + \nu\left[\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial\theta}\right]$$

$$u\frac{\partial w}{\partial r} + v\frac{\partial w}{r\partial\theta} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\nabla^2 w\right], \text{ where } \nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$$

Since motion is same for all θ, therefore

$$\frac{\partial p}{\partial\theta} = 0.$$

Now, in this case, the above equation reduce to

$$\frac{\partial v}{\partial \theta} = 0 \longrightarrow (i)$$

$$-\frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \longrightarrow (ii)$$

$$\nu\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} - \frac{v}{r^2}\right] = 0 \longrightarrow (iii)$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = 0 \longrightarrow (iv)$$

Now, $p = p(r)$ only

from (iii) $\quad \dfrac{d^2 v}{dr^2} + \dfrac{1}{r}\dfrac{dv}{\partial r} - \dfrac{v}{r^2} = 0 \longrightarrow (v)$

[Since motion is in the circle, therefore v is not a function of z.]

If we put, $v = \Omega r$, where Ω is a variable.

$$\frac{dv}{dr} = \Omega + r\frac{d\Omega}{dr}$$

$$\frac{d^2 v}{dr^2} = \frac{d\Omega}{dr} + \frac{d\Omega}{dr} + r\frac{d^2\Omega}{dr^2}$$

$$= 2\frac{d\Omega}{dr} + r\frac{d^2\Omega}{dr^2}$$

Substituting in (v)

$$2\frac{d\Omega}{dr} + r\frac{d^2\Omega}{dr^2} + \frac{\Omega}{r} + \frac{d\Omega}{dr} - \frac{\Omega r}{r^2} = 0$$

$$\Rightarrow r\frac{d^2\Omega}{dr^2} + 3\frac{d\Omega}{dr} = 0$$

Multiplying both sides by r^2, we get,

$$r^3\frac{d^2\Omega}{dr^2} + 3r^2\frac{d\Omega}{dr} = 0$$

$$\Rightarrow \frac{d}{dr}\left[r^3\frac{d\Omega}{dr}\right] = 0$$

$$\Rightarrow r^3 \cdot \frac{d\Omega}{dr} = \text{constant} = A \text{ (say)}$$

$$\Rightarrow \frac{d\Omega}{dr} = \frac{A}{r^3}$$

$$\Rightarrow \Omega = -\frac{A}{2r^2} + B$$

$$v = \Omega r = -\frac{A}{2r} + Br. \longrightarrow (vi)$$

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