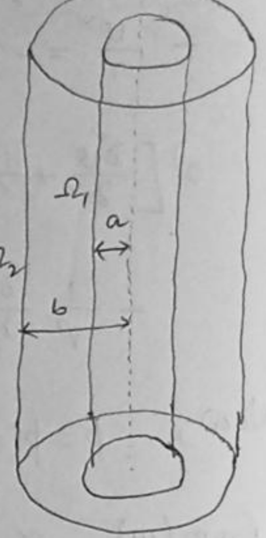
Rotation of fluid between long coaxial cylinders:



Let Ω_1 and Ω_2 be the angular velocities of the cylinders. Then v on

$r = a$ is $\Omega_1 a$.

and v on $r = b$ is $\Omega_2 b$.

But we have

$$v = Br - \frac{A}{2r}$$

$\therefore$ on $r = a$, $\Omega_1 a = Ba - \frac{A}{2a} \longrightarrow$ (vii)

and on $r = b$, $\Omega_2 b = Bb - \frac{A}{2a} \longrightarrow$ (viii)

Subtracting (viii) from (vii)

$$B(a^2 - b^2) = \Omega_1 a^2 - \Omega_2 b^2$$

$$\Rightarrow B = \frac{\Omega_1 a^2 - \Omega_2 b^2}{a^2 - b^2}$$

from (vii) and (viii) $\Rightarrow ab(\Omega_1 - \Omega_2) = A\left(\frac{a}{2b} - \frac{b}{2a}\right)$

$$\Rightarrow A \frac{a^2 - b^2}{2ab} = ab(\Omega_1 - \Omega_2)$$

$$\Rightarrow A = \frac{2a^2b^2(\Omega_1 - \Omega_2)}{a^2 - b^2}$$

$$\therefore v = \frac{\Omega_1 a^2 - \Omega_2 b^2}{a^2 - b^2} \cdot r - \frac{1}{2r} \cdot \frac{2a^2b^2(\Omega_1 - \Omega_2)}{a^2 - b^2} \rightarrow \text{(ix)}$$

If Ω_1 and Ω_2 are the same signs $v \neq 0$ in $a \leq r \leq b$.

If Ω_1 and Ω_2 are of opposite signs, then putting $-\Omega_2 = \Omega_2$

$$\therefore\ v = \frac{\Omega_1 a^2 + \Omega_2 b^2}{a^2 - b^2}\cdot r - \frac{a^2 b^2(\Omega_1 + \Omega_2)}{r\cdot(a^2 - b^2)}$$

$\therefore\ v = 0$ at

$$\frac{a^2\Omega_1 + \Omega_2 b^2}{a^2 - b^2}\cdot r - \frac{a^2 b^2(\Omega_1 + \Omega_2)}{r(a^2 - b^2)} = 0$$

$$\Rightarrow r^2(\Omega_1 a^2 + \Omega_2 b^2) = a^2 b^2(\Omega_1 + \Omega_2)$$

$$\Rightarrow r^2 = \frac{a^2 b^2(\Omega_1 + \Omega_2)}{(a^2\Omega_1 + b^2\Omega_2)}$$

$$\Rightarrow r = ab\sqrt{\frac{\Omega_1 + \Omega_2}{\Omega_1 a^2 + \Omega_2 b^2}}$$

Viscous stress on the inner cylinder is

$$\sigma_{r\theta} = \mu\left[\frac{\partial u}{r\partial\theta} - \frac{v}{r} + \frac{\partial v}{\partial r}\right]$$

$$= \mu\left[-\frac{\Omega r}{r} + \frac{\partial(\Omega r)}{\partial r}\right]$$

$$= \mu\left[-\Omega + \Omega + r\cdot\frac{\partial\Omega}{\partial r}\right]$$

$$= \mu r\cdot\frac{\partial\Omega}{\partial r}$$

$$= \mu r\frac{\partial}{\partial r}\left[-\frac{A}{2r^2} + B\right]$$

$$= \mu r\left[-\frac{A(-2)}{2r^3}\right]$$

$$\Rightarrow \sigma_{r\theta} = \frac{A\mu}{r^2}$$

$$\therefore \left[\sigma_{r\theta}\right]_{r=a} = \left[\frac{A\mu}{r^2}\right]_{r=a} = \frac{A\mu}{a^2}$$

$$= \frac{\mu}{a^2}\left[\frac{2a^2b^2(\Omega_1 - \Omega_2)}{a^2-b^2}\right]$$

$$= \frac{2\mu b^2(\Omega_1 - \Omega_2)}{(a^2-b^2)}$$

Total stress on $r=b$ is

$$\frac{A\mu}{b^2} = \frac{2\mu a^2(\Omega_1 - \Omega_2)}{(a^2-b^2)}$$

Again, if we put $b \to \infty$, $\Omega_2 = 0$ in (ix), then We have

$$V = \lim_{b\to\infty}\left[\frac{\Omega_1 a^2}{a^2-b^2}\cdot r - \frac{a^2b^2\Omega_1}{r(a^2-b^2)}\right]$$

$$= \lim_{b\to\infty}\left[\frac{\frac{\Omega_1 a^2}{b^2}}{\frac{a^2}{b^2}-1}\cdot r - \frac{a^2\Omega_1}{r\left(\frac{a^2}{b^2}-1\right)}\right]$$

$$= 0 - \frac{a^2\Omega_1}{r(0-1)}$$

$$\Rightarrow V = \frac{a^2\Omega_1}{r}$$

This is the velocity due ~~to~~ a line vortex at $r=0$.

$\therefore$ The cylinder behaves like a vortex filament of strength $a^2\Omega \times 2\pi = 2\pi a^2\Omega$.

#