Open sets and closed sets in a topological space:

Let (X, τ) be a topological space. A subset G of X is said to be open in X or an open set in X if $G \in \tau$ and a subset F of X is said to be closed in X or a closed set of X if F^c is open in X.

Ex: In a discrete topological space $(X, \mathcal{D})$, every subset of $(X, \mathcal{D})$ is open and hence every subset of X is called as well as closed.

Ex: In an indiscrete topological space $(X, \mathcal{I})$, the open sets are ϕ and X, these are also closed sets of X.

Closure of a set:

Let (X, τ) be a topological space and let A be a subset of X. The closure of A, denoted by $\bar{A}$, is defined as the intersection of all closed supersets of A.

Note: (i) $\bar{A} = \bigcap_{\alpha} F_\alpha$, where F_α is a closed set and $A \subseteq F_\alpha$.

(ii) $\bar{A}$ is the smallest closed set containing A.

(iii) $\bar{A} = A$ if and only if A is closed.

(iv) If F be any closed superset of A, then $A \subseteq \bar{A} \subseteq F$

Example: Let $X = \{a, b, c, d, e\}$

$T = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$

Then (X, T) is a topological space.

The closed set in X are,

X, $\{b, c, d, e\}$, $\{a, b, e\}$, $\{b, e\}$, $\{a\}$ and ϕ

Let $A = \{b, d\}$; Then the closed set containing A are X and $\{b, c, d, e\}$

$\therefore \overline{A} = X \cap \{b, c, d, e\} = \{b, c, d, e\}$

Let, $B = \{b, e\}$. Then the closed set containing B are X, $\{b, c, d, e\}$, $\{a, b, e\}$, $\{b, e\}$

$\therefore \overline{B} = X \cap \{b, c, d, e\} \cap \{a, b, e\} \cap \{b, e\}$

$= \{b, e\}$

$= B$

Let $C = \{c, e\}$.

$\therefore \overline{C} = X \cap \{b, c, d, e\} = \{b, c, d, e\}$

Example: In a discrete topological space (X, D)

$\overline{A} = A$, $\forall A \subseteq X$ since in (X, D) every subset of X is closed.

Example: In an indiscrete topological space $(X, \mathcal{J})$, since the only closed subset of X are ϕ and X,

$$\therefore \bar{A} = \begin{cases} \phi, & \text{if } A = \phi \\ X, & \text{if } A \neq \phi \end{cases}$$

Ex: Let T be the topology on the set of real numbers $\mathbb{R}$ consisting of ϕ, $\mathbb{R}$ and all open intervals of the form (a, ∞), $a \in \mathbb{R}$. Then the closed sets in $(\mathbb{R}, T)$ are $\mathbb{R}$, ϕ and all intervals of the form $(-\infty, a]$, $a \in \mathbb{R}$

Then,

$$\overline{[3, 12)} = (-\infty, 12]$$

$$\overline{\{3, 7, 14, 19, 47\}} = (-\infty, 47]$$

$$\overline{\{5, 10, 15, 20, \cdots\}} = (-\infty, \infty) = \mathbb{R}$$

Theorem on closure:

Theorem: Let (X, T) be a topological space. If A and B are any two subsets of X, then,

(i) $\bar{\phi} = \phi$

(ii) $A \subseteq \bar{A}$

(iii) $\bar{\bar{A}} = \bar{A}$

(iv) $\overline{(A \cup B)} = \bar{A} \cup \bar{B}$

(v) $A \subset B \Rightarrow \bar{A} \subset \bar{B}$

proof:

(i) Since ϕ is a closed set, therefore $\bar{\phi} = \phi$

(ii) Since $\bar{A}$ is the smallest closed superset of A

$\therefore \bar{A}$ is a superset of A

$\therefore \quad A \subseteq \bar{A}$

(iii) Since $\bar{A}$ is a closed set

$\therefore \bar{\bar{A}} = \bar{A}$

(iv) Since $A \subseteq \bar{A}$ and $B \subseteq \bar{B}$

$\therefore \quad A \cup B \subseteq \bar{A} \cup \bar{B}$

Again $\bar{A}$ and $\bar{B}$ are closed sets

$\therefore \bar{A} \cup \bar{B}$ is also a closed set

$\therefore \quad \overline{A \cup B} \subseteq \bar{A} \cup \bar{B} \quad \rightarrow (i)$

Also $A \subseteq A \cup B$ and $B \subseteq A \cup B$

and ~~$\bar{A} \subseteq \overline{A \cup B}$~~ $A \cup B \subseteq \overline{A \cup B}$

$\therefore \quad A \subseteq \overline{A \cup B}$ and $B \subseteq \overline{A \cup B}$

$\Rightarrow \quad \bar{A} \subseteq \overline{A \cup B}$ and $\bar{B} \subseteq \overline{A \cup B}$

$\Rightarrow \quad \bar{A} \cup \bar{B} \subseteq \overline{A \cup B} \quad \rightarrow (ii)$

$\therefore$ (i) and (ii) $\Rightarrow \quad \overline{A \cup B} = \bar{A} \cup \bar{B}$

(v) Since $A \subseteq B$, therefore $A \cup B = B$

$\therefore \overline{A \cup B} = \overline{B}$

$\Rightarrow \overline{A} \cup \overline{B} = \overline{B}$

$\Rightarrow \overline{A} \subseteq \overline{B}$.

<u>Theorem</u>: Let Y be a subspace of a topological space (X, τ) and A be a subset of Y. Then the closure of A in Y equals $\overline{A} \cap Y$.

<u>Pf</u>: Since $A \subset Y$ and $Y \subset X$

$\therefore A \subset X \Rightarrow \overline{A}$ is a closed set in X.

Since $\overline{A} \cap Y$ is closed in Y and $A \subset Y$, $A \subset \overline{A}$

$\therefore A \subset \overline{A} \cap Y$

$\therefore \overline{A} \cap Y$ is a closed superset of A in Y.

Now let $F \cap Y$ be any other closed superset of A in Y. Then F is a closed superset of A in X.

$\therefore A \subset \overline{A} \subset F$

$\Rightarrow A \cap Y \subseteq \overline{A} \cap Y \subseteq F \cap Y$

$\Rightarrow A \subset \overline{A} \cap Y \subseteq F \cap Y$

This shows that $F \cap Y$ is the smallest closed superset of A in Y. Hence the closure of A in Y equals $\overline{A} \cap Y$.

H.W

Q) Let T be a topology on a set X consisting of four sets, i.e $T = \{\phi, X, A, B\}$, where A and B are non empty distinct proper subset of X. What condition must A and B satisfy.

Q) Consider the following topology on $X = \{a, b, c, d, e\}$

$T = \{\phi, X, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, e\}\}$

(i) List the closed subset of X

(ii) Determine the closure of the sets $\{a\}$, $\{b\}$ and $\{c, e\}$

Q) Let T be a topology on $\mathbb{N}$ which consists of ϕ and all subset of $\mathbb{N}$ of the form

$E_n = \{n, n+1, n+2, \ldots\}$, where $n \in \mathbb{N}$.

(i) Determine the closed subsets of $(\mathbb{N}, T)$

(ii) Determine the closure of the sets $\{7, 24, 47, 85\}$ and $\{3, 6, 9, 12\}$.