$\boxed{\text{Series Solution}}$

in

Homogeneous second order linear Differential Equation with variable coefficients.

$$\boxed{P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)\,y = 0} \quad \text{— (1)}$$

Here:-

$P_0(x)$, $P_1(x)$ & $P_2(x)$ → Polynomials in Powers of x.

Homogeneous → R.H.S zero → No terms of x only.

Second order → Second Derivative

Linear DE → Max Power of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ → one.

Variable coefficients → $\because$ P_0, P_1 & P_2 are Polynomials in x

From Eq. (1):-

$$\frac{d^2y}{dx^2} + \frac{P_1(x)}{P_0(x)}\frac{dy}{dx} + \frac{P_2(x)}{P_0(x)}y = 0$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0} \quad \text{--- (2)}$$

Eq. (2) is called Normal form / Canonical form / Standard form of Homogeneous Second order Linear DE with Constant Coefficients.

Here:-

$$P(x) = \frac{P_1(x)}{P_0(x)}$$

$$Q(x) = \frac{P_2(x)}{P_0(x)}$$

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Classification of Point $x = a$:-

Point: $x = a$

Ordinary Point (OP)

If :- at $x = a$

$P_0(a) \neq 0$

Singular Point (SP)

If :- at $x = a$

$P_0(a) = 0$

Regular Singular Point (RSP)

If :- both

$$\lim_{x \to a} (x-a) \frac{P_1(x)}{P_0(x)} = \text{Finite}$$

$$\lim_{x \to a} (x-a)^2 \frac{P_2(x)}{P_0(x)} = \text{Finite}$$

Irregular Singular Point (ISP)

Otherwise

Note:-

① Finite or Analytic

② $\frac{P_1(x)}{P_0(x)} = P(x)$

③ $\frac{P_2(x)}{P_0(x)} = Q(x)$

① **Ordinary Point (OP) :-**

Point $x = a$ is said to be an ordinary Point iff at $x = a$, $P_0(a) \neq 0$

$\boxed{OR}$

Point $x = a$ is said to be an ordinary Point if both $\frac{P_1(x)}{P_0(x)}$ and $\frac{P_2(x)}{P_1(x)}$ are Finite (Analytic) at $x = a$.

② **Singular Point (SP) :-**

Point $x = a$ is said to be a singular Point iff at $x = a$, $P_0(a) = 0$

③ **Regular Singular Point (RSP) :-**

A Singular Point $x = a$ is said to be Regular iff both

$\lim_{x \to a} (x-a) \frac{P_1(x)}{P_0(x)}$ & $\lim_{x \to a} (x-a)^2 \frac{P_2(x)}{P_0(x)}$ are finite (Analytic)

OR

$\lim_{x \to a} (x-a) P(x)$ & $\lim_{x \to a} (x-a)^2 Q(x)$ are finite (Analytic)

④ **Irregular Singular Point (ISP) :-**

A singular point $x=a$ which is not regular is called as Irregular Singular Point.

Q.1 Find the Ordinary Point, Singular Point(s), Regular singular and Irregular singular point(s) of the Differential Eq.

$$x^3(x-1)\frac{d^2y}{dx^2}+(x-1)\frac{dy}{dx}+4xy=0$$

Sol:- From the given Diff. Eq.

$P_0(x) = x^3(x-1)$, $P_1(x) = x-1$

$P_2(x) = 4x$

Now, for Ordinary Point (s) :-

$P_0(a) \neq 0$

$a^3(a-1) \neq 0$

$a^3 \neq 0$	$a-1 \neq 0$
$a \neq 0$	$a \neq 1$

So Every value of x is an Ordinary Point except $x=0$, $x=1$

For Singular Point:-

$P_0(a) = 0 \Rightarrow a^3(a-1) = 0$

$a^3 = 0$	$a-1=0$
$a=0$	$a=1$

Singular Points $x=0$, $x=1$

At Point $a=0$:-

$$\underset{x\to a}{Lt}\ (x-a)\frac{P_1(x)}{P_0(x)}$$

$$\underset{x\to 0}{Lt}\ \cancel{(x-0)}\frac{\cancel{(x-1)}}{x^{\cancel{3}\,2}\cancel{(x-1)}} = \underset{x\to 0}{Lt}\ \frac{1}{x^2}$$

$$= \infty$$

= Infinite = Non-Analytic

So $x=0$ is irregular singular Point

At Point $x=1$

$$\underset{x\to 1}{Lt}\ \cancel{(x-1)}\frac{(x-1)}{x^3\cancel{(x-1)}} = \frac{0}{1} = 0$$

Finite

$$\underset{x\to a}{Lt}\ (x-a)^2\frac{P_2(x)}{P_0(x)}$$

$$\underset{x\to 1}{Lt}\ \cancel{(x-1)}^{2}\ \frac{4x}{x\,x^3\cancel{(x-1)}} = \frac{0}{1}$$

= Finite

So $x=1$ Regular singular Point

Some Important Differential Equations :-

① **Legendre's Differential Equation**

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

$x=0$ is ordinary Point, n = real No.

② **Bessel's Differential Equation**

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0$$

n = Real constant = Order of Bessel's DE

$x=0$ is Regular singular Point.

③ **Chebyshev Differential Eq.**

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0$$

$x=0$ is Ordinary Point, n = real No.

Power Series Method

1. Applied when $x = a$ is an Ordinary Point of given second order Diff. Eq.

2. Complete Solution :-

$$y = \sum_{n=0}^{\infty} a_n (x-a)^n$$

if $a = 0$:-

$$y = \sum_{n=0}^{\infty} a_n x^n$$

3. Applicable in Legendre's and Chebyshev Differential Equations [$\because x = 0$ Ordinary Point]

Frobenius Method

1. Applied when $x = 0$ is a Regular Singular Point of given second order Diff. Eq.

2. One of the Solution

$$y = \sum_{n=0}^{\infty} a_n x^{m+n}$$

- m can have any value.
- n whole Number
- a_n constant coefficients.

3. Applicable in Bessel's Diffi. Equation of order n. [$\because x = 0$ Regular Singular Point]

Power Series Method	Frobenius Method
4. Final solution contains only two constants a_0 and a_1, from all possible values of a_n	4. Final solution contains only one constant a_0, from all possible values of a_n.
5. Single solution case in output (Final solution)	5. four types of sub cases can be there in writing final solution, depending on the values of m (i.e m_1 & m_2)

Point $x = 0$
- → Ordinary Point → Power Series Method
- → Regular Singular Point → Frobenius Method
- → Irregular Singular Point → Solution Does not Ex

Power Series method of Series Solution (when $x = a$ is Ordinary Point) 9

FLOW CHART:-

Homogeneous second order Linear Diff. Eq. with variable coffi.
(Assume $x=0$ is an Ordinary Point)

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Assume the solution $y = \sum_{n=0}^{\infty} a_n x^n$ and find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2} = ?$

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Substitute the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in given Diff. Eq.

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Make all the powers of x to x^n, by applying Indexing and make all the starting values of n same (by expanding).

↓

Equate the Extra (if any) powers of x (outside summation) to zero, to find starting values of coefficient a_n in terms of a_0 and a_1

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Now Equate the coefficient of x^n to zero, we will get a Recurrance Relation / Difference Equation.

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Take different values of n in the Recurrance Relation to determine various values of a's in the terms of a_0 & a_1

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Substitute the values of a_2, a_3, a_4 ...... in assumed solution y, to get series solution of given Differential Equation in terms of a_0 and a_1.