So substitute the values of $\frac{d^2y}{dx^2}$ & y in eq. (1), we get:-

$$\left[\sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2}\right] + x\left[\sum_{n=0}^{\infty} a_n x^n\right] = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2} + \sum_{n=0}^{\infty} a_n\, x^n \cdot x = 0$$

$$\Rightarrow \boxed{\sum_{n=2}^{\infty} n(n-1)\, a_n\, \underline{x}^{n-2} + \sum_{n=0}^{\infty} a_n\, \underline{x}^{n+1} = 0}$$

Imp. $\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n + \sum_{n=1}^{\infty} a_{n-1}\, x^n = 0$ [By Applying Indexing]

Note Relation ? Note

$$\Rightarrow \left[(2)(1)\, a_2\, x^0 + \sum_{n=1}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n\right] + \sum_{n=1}^{\infty} a_{n-1}\, x^n = 0$$

by Expanding

(14)

$$\Rightarrow 2a_2x^0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_{n-1}\right]x^n = 0 \quad -(2)$$

Now by Equating coeffi. of x^0 to zero, we get:-

$$2a_2 = 0$$

$$\boxed{a_2 = 0}$$

Now by Equating coeffi. of x^n to zero, we get:-

$$\left[(n+2)(n+1)a_{n+2} + a_{n-1}\right] = 0 \quad \text{for all } n \geq 1$$

$$(n+2)(n+1)a_{n+2} = -a_{n-1}$$

$$\boxed{a_{n+2} = \frac{-a_{n-1}}{(n+2)(n+1)}} \quad \text{for all } n \geq 1$$

which is the required Recurrance Relation/Difference Eq.

Now for $n=1$:- $a_3 = \frac{-a_0}{(3)(2)} \Rightarrow \boxed{a_3 = -\frac{a_0}{6}}$

(15)

For $n=2$:- $a_4 = -\frac{a_1}{(4)(3)} \Rightarrow \boxed{a_4 = -\frac{a_1}{12}}$

For $n=3$:- $a_5 = \frac{-a_2}{(5)(4)} \Rightarrow a_5 = -\frac{a_2}{20} \Rightarrow \boxed{a_5 = 0}$

For $n=4$:- $a_6 = \frac{-a_3}{(6)(5)} \Rightarrow a_6 = \frac{-a_3}{30} = -\frac{1}{30}\left(-\frac{a_0}{6}\right) = \boxed{\frac{a_0}{180}}$

. . . .

Now, put the values of a_2, a_3, a_4, a_5 & a_6 in assumed solution, we get –

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

$$= a_0 + a_1 x + (0)x^2 + \left(-\frac{a_0}{6}\right)x^3 + \left(-\frac{a_1}{12}\right)x^4 + (0)x^5 + \left(\frac{a_0}{180}\right)x^6 + \cdots$$

$$\boxed{y = a_0\left(1 - \frac{x^3}{6} + \frac{x^6}{180} + \cdots \infty\right) + a_1\left(x - \frac{x^4}{12} + \cdots \infty\right)}$$ Ans

Q.3 Solve : $(1-x^2)\,y'' - xy' + 4y = 0$ in Series (16)

OR [UPTU, U.P – 2014, 2012, 2006]

Note Solve in series Chebyshev's Differential Eq. (when $n=2$)

OR

Solve the Differential Equation $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 4y = 0$

about the point $x=0$.

Note?

OR

Solve the Differential Equation $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 4y = 0$

near the point $x=0$.

Note

OR

Note

Find Power series solution of the Differential Equation

$(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 4y = 0$ about the ordinary Point $x=0$. Note?

(1)

Q.2 Solve in Series the Differential Equation

$$\frac{d^2y}{dx^2} + xy = 0$$

OR

Solve: $y'' + xy = 0$ in series.

[GTU, Gujrat – 2019, 2014, 2013
UPTU, U.P – 2010]

Sol:- According to the given Differential Equation :-

$$\boxed{\frac{d^2y}{dx^2} + xy = 0} \quad \text{— (1)}$$

The given Diff. Eq. (1) is Homogeneous Second Order Linear Differential Equation with variabe coefficients.

Now by comparing given diff. Eq. (1) with :-

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0$$

We get :- $P_0(x) = 1$, $P_1(x) = 0$, $P_2(x) = x$

$\boxed{\text{Now at Point } x=0}$:- $P_0(0) = 1 \neq 0$

Note?

↳ $x=0$ is an Ordinary Point.

⇓

Power Series Method ✓

Now let the solution of eq. ① is :-

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \text{———}$$

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} a_n \left(n x^{n-1}\right) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

↳ Note

$$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} (n a_n)(n-1) x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

↳ Note