$$g'_{\alpha\beta} = g_{ij} A^i_{(\alpha)} A^j_{(\beta)}$$

$$\therefore \frac{\partial g'_{\alpha\beta}}{\partial x^k} = \frac{\partial g_{ij}}{\partial x^k} A^i_{(\alpha)} A^j_{(\beta)} + g_{ij} \frac{\partial A^i_{(\alpha)}}{\partial x^k} A^j_{(\beta)} + g_{ij} A^i_{(\alpha)} \frac{\partial A^j_{(\beta)}}{\partial x^k}$$

Using (1) we get

$$\frac{\partial g'_{\alpha\beta}}{\partial x^k} = \frac{\partial g_{ij}}{\partial x^k} A^i_{(\alpha)} A^j_{(\beta)} + g_{ij} \left(-\Gamma^i_{kl} A^l_{(\alpha)}\right) A^j_{(\beta)} + g_{ij} A^i_{(\alpha)} \left(-\Gamma^j_{kl} A^l_{(\beta)}\right)$$

$$= \frac{\partial g_{ij}}{\partial x^k} A^i_{(\alpha)} A^j_{(\beta)} - g_{lj} \Gamma^l_{ki} A^i_{(\alpha)} A^j_{(\beta)} - g_{ij} \Gamma^l_{kj} A^i_{(\alpha)} A^j_{(\beta)}$$

$$= \left(\frac{\partial g_{ij}}{\partial x^k} - g_{lj} \Gamma^l_{ki} - g_{il} \Gamma^l_{kj}\right) A^i_{(\alpha)} A^j_{(\alpha)}$$

$$= \left(\frac{\partial g_{ij}}{\partial x^k} - \Gamma_{ki,j} - \Gamma_{kj,i}\right) A^i_{(\alpha)} A^j_{(\alpha)}$$

$$= 0 \qquad \left[\because \Gamma_{ki,j} + \Gamma_{kj,i} = \frac{\partial g_{ij}}{\partial x^k}\right.$$

$$\therefore \frac{\partial}{\partial x^k}\left(g'_{\alpha\beta}\right) = 0$$

This shows that $g'_{\alpha\beta}$ are constants throughout the region of space-time.

Thus we can find a coordinate system in the space time for which the ~~fund~~ fundamental tensors are constant. Hence the space-time is flat.

complete

[signature]

Conversely, let the Riemann-christoffel curvature tensor vanish in a space-time. Then it is possible to construct a uniform vector field by parallel displacement of a vector all over the region of space-time.

Let $A^i_{(\alpha)}$ be four uniform vector fields given by $\alpha = 1, 2, 3, 4$ where α is not a tensor suffix. Then the vanishing of Riemann-christoffel curvature tensor implies that the differential equation $A^i_{(\alpha),j} = 0$ is integrable.

Also $A^i_{(\alpha),j} = 0$

$$\Rightarrow \frac{\partial A^i_{(\alpha)}}{\partial x^j} + \Gamma^i_{jk} A^k_{(\alpha)} = 0$$

$$\Rightarrow \frac{\partial A^i_{(\alpha)}}{\partial x^j} = -\Gamma^i_{jk} A^k_{(\alpha)} \longrightarrow (1)$$

Now let us consider a coordinate system x'^i related to the current system x^i by the relation

$$dx^i = A^i_{(\alpha)} dx'^\alpha, \quad \alpha = 1, 2, 3, 4 \longrightarrow (2)$$

Since ds^2 is an invariant, therefore

$$g'_{\alpha\beta} dx'^\alpha dx'^\beta = g_{ij} dx^i dx^j$$

$$\Rightarrow g'_{\alpha\beta} dx'^\alpha dx'^\beta = g_{ij} (A^i_{(\alpha)} dx'^\alpha)(A^j_{(\beta)} dx'^\beta)$$

$$\Rightarrow g'_{\alpha\beta} dx'^\alpha dx'^\beta = g_{ij} A^i_{(\alpha)} A^j_{(\beta)} dx'^\alpha dx'^\beta$$

$$\Rightarrow (g'_{\alpha\beta} - g_{ij} A^i_{(\alpha)} A^j_{(\beta)}) dx'^\alpha dx'^\beta = 0$$

Since x'^α, x'^β being arbitrary, we must have