COMPLEX VARIABLE - II

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Derivatives

*If f (z) is a single-valued in some region **C** of the z plane, the deriv defined as:*

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

provided that the limit exists independent of the manner in which

In such case we say that f (z) is differentiable at z.

Analytic Functions

*If the derivative of f (z) exists at all points of a region **C** of the z p said to be analytic in **C**.*

analytic = regular = holomorphic

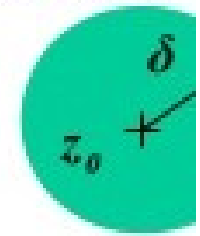
A function f (z) is said to be analytic at a point z_0 if the ts a $|z-z_0| < \delta$ in which f ' (z) exists.

Singular Points

A point at which f (z) is not analytic is called a singular point. There
of singular points:

1. Isolated Singularity

The point z_0 at which f (z) is not analytic is called an isolated singula
a neighborhood of z_0 in which there are not singular points.
If no such a neighborhood of z_0 can be found then we call z_0 a non-isolated singular point.



2. Poles

If we can find a positive integer n such that $\lim_{z \to z_0} (z - z_0)^n f(z) = A$
and $\varphi(z) = (z - z_0)^n f(z)$ *is analytic at $z = z_0$*
then $z = z_0$ is called a pole of order n. If n = 1, z is called a simple poi

Example $f(z) = \dfrac{(4z+3)(8z+1)}{\text{[illegible]}}$

Cauchy-R and nn Equ . tions

A necessary (but not sufficient) condition that f (z) = u (x,y) +i v (x
analytic in a region C, is that u and v satisfy the Cauchy-Riemann e

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Proof:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Provided that the limit exists independent of the manner in which $\Delta z \to 0$.

Choose $\Delta z = \Delta x$ $\longrightarrow \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

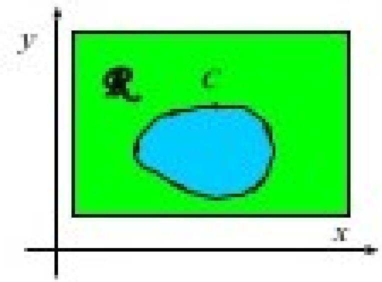
Choose $\Delta z = i\Delta y$ $\longrightarrow \quad f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

Equalizing those two expressions we obtain:

Cauchy's Theorem

If f (z) is analytic with derivative f ' (z) which is continuous at a
and on a simple closed curve C, then:

$$\oint_C f(z)\,dz = 0$$



Proof:

Since $f(z) = u(x,y) + i\,v(x,y)$ **is analytic and has continuous first order derivative**

$$f'(z) = \frac{d\,f}{d\,z} \overset{z=x}{=} \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \overset{z=iy}{=} \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$ ***Cauchy - Riemann***

$$\oint f(z)\,dz = \oint (u + i\,v)(dx + i\,dy) = \oint (u\,dx - v$$

Cauchy's Integral Formulas and Related Theorems

Cauchy's Integral Formulas

If f (z) is analytic inside and on a simple closed curve C and a is any point inside C then

$$f(a)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

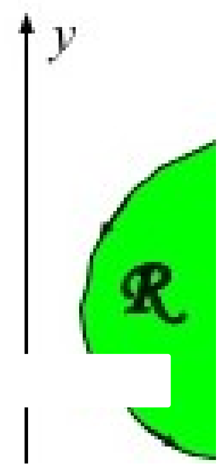
Proof:

Let chose a circle Γ with center at a

$$\Gamma=\{z : z = a+\varepsilon e^{i\theta},\ \theta\in[0,2\pi]\}$$

Since f (z)/ (z-a) is analytic in the region defined between C and the circle Γ we n use:

$$\oint_C \frac{f(z)}{z-a}\,dz=\oint_\Gamma \frac{f(z)}{z-a}\,dz$$

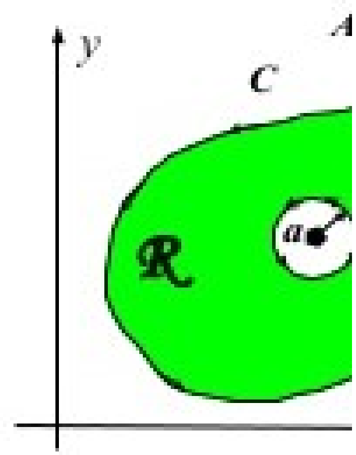


Cauchy's Integral Formulas for the n Derivative of a Functio

If f (z) is analytic inside and on a simple closed curve C and a is any point inside C, where the n derivative exists, then

$$f^{(n)}(a)=\frac{n!}{2\pi i}\oint_{c}\frac{f(z)}{(z-a)^{n+1}}\,dz$$

Proof:



Let prove this by induction.

For n = 0 we found $f(a)=\frac{1}{2\pi i}\int_{c}\frac{f(z)}{z-a}\,dz$

Assume that this is true for n-1:

$$f^{(n-1)}(a)=\frac{(n-1)!}{2\pi i}\oint_{c}\frac{f(z)}{(z-a)^{n}}\,dz$$

Then we can differentiate under the sign of integration:

Thanks