Ex. Establish the formula $\frac{1}{8}\frac{\pi a^4}{\mu L}(P_1 - P_2)$ for the rate of steady flow of an incompressible liquid through a uniform circular pipe of radius 'a', P_1 and P_2 being the pressure of two section of the pipe distant 'l' apart.

Soln. We consider the steady flow of an incompressible viscous fluid through a circular pipe of radius 'a'. Let u, v, w be the velocity components in the direction of r, θ, z respectively. Then the boundary conditions are

$$\left.\begin{array}{l} u=0,\ v=0,\ w=0 \quad \text{at } r=a \\ u=0,\ v=0,\ \frac{dw}{dr}=0 \quad \text{at } r=0 \end{array}\right\} \longrightarrow \text{(i)}$$

the axis of a pipe is taken as z-axis. In view of these conditions we take

$$u=0,\ v=0,\ w=w(r).$$

Now, the equation of continuity and motion are

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial v}{r\partial\theta}+\frac{\partial w}{\partial z}=0$$

$$u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial\theta}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left[\nabla^2 u-\frac{2}{r^2}\frac{\partial v}{\partial\theta}-\frac{u}{r^2}\right]$$

$$u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial\theta}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=-\frac{1}{\rho}\frac{\partial p}{r\partial\theta}+\nu\left[\nabla^2 v+\frac{2}{r^2}\frac{\partial u}{\partial\theta}-\frac{v}{r^2}\right]$$

$$u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial\theta}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w,$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}$

In this case, the equation reduces to

$$\frac{\partial w}{\partial z}=0$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = 0$$

$$-\frac{1}{\rho}\frac{\partial p}{r\partial \theta} = 0$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right] = 0$$

i.e. the equation of motion in the direction of r and θ gives that the hydrostatic pressure p is independent of r and θ and the equation of motion in the z-direction is given by

$$\mu\left[\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right] = \frac{dp}{dz} \longrightarrow \text{(ii)}$$

The left side of this equation is a function of r only and the right side is a function of z-only, hence both must be a constant.

From (ii) $\frac{d}{dr}\left[r\frac{dw}{dr}\right] = \frac{r}{\mu}\frac{dp}{dz}$

Integrating, $r\frac{dw}{dr} = \frac{1}{\mu}\frac{dp}{dz}\frac{r^2}{2} + B \longrightarrow \text{(iii)}$

$$\Rightarrow \frac{dw}{dr} = \frac{1}{2\mu}\frac{dp}{dz}r + \frac{B}{r}$$

Again, integrating,

$$w = \frac{1}{2\mu}\frac{dp}{dz}\frac{r^2}{2} + B\log r + C \longrightarrow \text{(iv)}$$

At $r=0$

(iii) $\Rightarrow 0 = 0 + B \Rightarrow B = 0$

At $r=0$

(iv) $\Rightarrow 0 = \frac{1}{2\mu}\frac{\partial p}{\partial z}\cdot\frac{a^2}{2} + B\log a + C$

$$\Rightarrow C = -\frac{a^2}{4\mu}\frac{dp}{dz}.$$

$$\therefore\ w = \frac{1}{2\mu}\frac{dp}{dz}\frac{r^2}{2} - \frac{a^2}{4\mu}\frac{dp}{dz}$$

$$= \frac{1}{4\mu}\frac{dp}{dz}(r^2 - a^2)$$

The maximum velocity is on the axis and is given by

$$w = \frac{a^2}{4\mu}\left(-\frac{dp}{dz}\right)$$

$$\bar{w} = \frac{w}{2}$$

The mean velocity is given by

$$\bar{w} = \frac{a^2}{8\mu}\left(-\frac{dp}{dz}\right)$$

And the volume of fluid flowing through the pipe per unit time is given by

$$Q = \pi a^2 \bar{w} = \frac{\pi a^4}{8\mu}\left(-\frac{dp}{dz}\right)$$

If the length of the pipe is 'l' and the pressure at the ends be P_1 and P_2 then,

$$-\frac{dp}{dz} = \frac{P_1 - P_2}{l}$$

and then the flux through the pipe is

$$Q = \frac{\pi a^4}{8l\mu}(P_1 - P_2).$$ #

[This flow is known as Hagan - Poiseuille flow].

Ex. Show that in the two-dimensional motion of a viscous liquid, acted on by a conservative system of forces the stream function satisfies the equation

$$\left(\nu\nabla^2 - \frac{\partial}{\partial t}\right)\nabla^2\psi = \frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)}$$

where ν is the kinematic viscosity.

Soln.

The Navier-Stokes equation for incompressible viscous fluid (under the action of conservative system of forces) is given by

$$\frac{\partial\vec{\xi}}{\partial t} + (\vec{q}\cdot\vec{\nabla})\vec{\xi} - (\vec{\xi}\cdot\vec{\nabla})\vec{q} = \nu\nabla^2\vec{\xi} \longrightarrow (1)$$

where $\vec{\xi} = \text{curl}\,\vec{q}$

Since the motion is two-dimensional motion of a viscous fluid, therefore

$$\vec{q} = (u, v, 0), \quad \vec{\xi} = (0, 0, \vec{\zeta})$$

$$\vec{\xi} = \text{Curl}\,\vec{q} = \vec{\nabla}\times\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & 0 \end{vmatrix}$$

Now from (i) we have,

$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\zeta = \nu\nabla^2\zeta - \frac{\partial\zeta}{\partial t}$$

$$= \left(\nu\nabla^2 - \frac{\partial}{\partial t}\right)\zeta \longrightarrow (ii)$$

Now, if the stream function ψ exists then

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x}$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \nabla^2\psi$$

$\therefore$ (ii) can be written as

$$\left(\nu\nabla^2 - \frac{\partial}{\partial t}\right)\nabla^2\psi = u\left(\frac{\partial \zeta}{\partial x}\right) + v\left(\frac{\partial \zeta}{\partial y}\right)$$

$$= \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\left(\nabla^2\psi\right) - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\left(\nabla^2\psi\right)$$

$$= \frac{\partial}{\partial x}(\psi)\frac{\partial}{\partial y}\left(\nabla^2\psi\right) - \frac{\partial}{\partial y}(\psi)\frac{\partial}{\partial x}\left(\nabla^2\psi\right)$$

$$= \frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)}$$

#

<u>Ex.</u> A Liquid occupying the space between two coaxial circular cylinder is acted upon by a force $\frac{c}{r}$ per unit mass, where r is the distance from the axis, the lines of force being circles round the axis. Prove that in the steady motion the velocity at any point is given by the formula

$$\frac{1}{2}\frac{c}{\nu}\left[\frac{b^2}{r}\cdot\frac{(r^2-a^2)}{(b^2-a^2)}\log\frac{b}{a} - r\cdot\log\frac{r}{a}\right]$$

where ν is the coefficient of kinematic viscosity and a, b are the two radii.

<u>Soln.</u> Let z-axis be the axis of the cylinders. If u, v, w be the velocity components along r, θ, z directions.

Then $u = w = 0$, $v = r\Omega$ $\longrightarrow$ (i)

where Ω is the angular velocity of the liquid at any point.