$\therefore$ (ii) can be written as

$$\left(\nu\nabla^2 - \frac{\partial}{\partial t}\right)\nabla^2\psi = u\left(\frac{\partial \zeta}{\partial x}\right) + v\left(\frac{\partial \zeta}{\partial y}\right)$$

$$= \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\left(\nabla^2\psi\right) - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\left(\nabla^2\psi\right)$$

$$= \frac{\partial}{\partial x}(\psi)\,\frac{\partial}{\partial y}\left(\nabla^2\psi\right) - \frac{\partial}{\partial y}(\psi)\,\frac{\partial}{\partial x}\left(\nabla^2\psi\right)$$

$$= \frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)}$$

#

Ex. A liquid occupying the space between two coaxial circular cylinder is acted upon by a force $\frac{c}{r}$ per unit mass, where r is the distance from the axis, the lines of force being circles round the axis. Prove that in the steady motion the velocity at any point is given by the formula

$$\frac{1}{2}\,\frac{c}{\nu}\left[\frac{b^2}{r}\cdot\frac{(r^2-a^2)}{(b^2-a^2)}\log\frac{b}{a} - r\cdot\log\frac{r}{a}\right]$$

where ν is the coefficient of kinematic viscosity and a, b are the two radii.

Soln. Let z-axis be the axis of the cylinders. If u, v, w be the velocity components along r, θ, z directions.

Then $u = w = 0$, $v = r\Omega \longrightarrow (i)$

where Ω is the angular velocity of the liquid at any point.

Now, the equation of continuity and motion are

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{r\partial \theta} + \frac{\partial w}{\partial z} = 0$$

$$u\frac{\partial u}{\partial r} + v\frac{\partial u}{r\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\nabla^2 u - \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{u}{r^2}\right]$$

$$u\frac{\partial v}{\partial r} + v\frac{\partial v}{r\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{\rho}\frac{\partial p}{r\partial \theta} + \nu\left[\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta}\right]$$

$$u\frac{\partial w}{\partial r} + v\frac{\partial w}{r\partial \theta} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\nabla^2 w\right)$$

In this case, the equations reduce to

$$\frac{\partial v}{\partial \theta} = 0$$

$$-\frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\nu\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} - \frac{v}{r^2}\right] = 0,$$

$\frac{\partial p}{\partial \theta} = 0$, since motion is same for all θ.

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = 0$$

Now, the equation of motion in the θ direction gives,

$$u\frac{\partial v}{\partial r} + v\frac{\partial v}{r\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = \frac{c}{r} - \frac{1}{\rho}\frac{\partial p}{r\partial \theta} + \nu\left[\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta}\right]$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$

In this case, this equation reduces to

$$0 = \frac{c}{r} + \nu\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right] \longrightarrow (ii)$$

$$\Rightarrow \frac{c}{r} + \nu\left[\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2}\right] = 0 \rightarrow (iii);$$

from equation of continuity we have $\frac{\partial v}{r\partial \theta} = 0$

Now, $v = r\Omega$

$$\frac{dv}{dr} = \Omega + r\frac{d\Omega}{dr}$$

$$\frac{d^2v}{dr^2} = \frac{d}{dr}\left[\Omega + r\frac{d\Omega}{dr}\right]$$

$$= 2\frac{d\Omega}{dr} + r\frac{d^2\Omega}{dr^2}$$

From (iii) we have,

$$\frac{c}{r} + \nu\left[r\frac{d^2\Omega}{dr^2} + 3\frac{d\Omega}{dr}\right] = 0$$

Multiplying by r^2

$$cr + \nu\left[r^3\frac{d^2\Omega}{dr^2} + 3r^2\frac{d\Omega}{dr}\right] = 0$$

$$\Rightarrow \frac{d}{dr}\left[r^3\frac{d\Omega}{dr}\right] = -\frac{cr}{\nu}$$

Integrating, $r^3\frac{d\Omega}{dr} = -\frac{cr^2}{2\nu} + A$

$$\Rightarrow \frac{d\Omega}{dr} = -\frac{c}{2\nu r} + \frac{A}{r^3}$$

Again, integrating,

$$\Omega = -\frac{c}{2\nu}\log r - \frac{A}{2r^2} + B \longrightarrow \text{(iv)}$$

Boundary conditions are

$\Omega = 0$ on $r = a$

$\Omega = 0$ on $r = b$.

when $r = a$, from (iv)

$$0 = -\frac{c}{2\nu}\log a - \frac{A}{2a^2} + B$$

when $r = b$, from (iv)

$$0 = -\frac{c}{2\nu}\log b - \frac{A}{2b^2} + B$$

Substracting, we get,

$$\frac{A}{2}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)+\frac{c}{2\nu}\left(\log b-\log a\right)=0$$

$$\Rightarrow A\left(\frac{a^2-b^2}{a^2b^2}\right)+\frac{c}{\nu}\log\frac{b}{a}=0$$

$$\Rightarrow A=\frac{c\,a^2b^2}{\nu(a^2-b^2)}\log\frac{a}{b}$$

$$\therefore B=\frac{A}{2a^2}+\frac{c}{2\nu}\log a$$

$$\Rightarrow B=\frac{1}{2a^2}\,\frac{c\,a^2b^2}{\nu(a^2-b^2)}\log\frac{a}{b}+\frac{c}{2\nu}\log a$$

$$\therefore \Omega=-\frac{c}{2\nu}\log r-\frac{A}{2r^2}+\frac{A}{2a^2}+\frac{c}{2\nu}\log a$$

$$=-\frac{c}{2\nu}\log\frac{r}{a}+\frac{A}{2}\left(\frac{1}{a^2}-\frac{1}{r^2}\right)$$

$$\Rightarrow \Omega=-\frac{c}{2\nu}\log\frac{r}{a}+\frac{1}{2}\,\frac{c\,a^2b^2}{\nu(b^2-a^2)}\log\frac{b}{a}\times\left(\frac{1}{a^2}-\frac{1}{r^2}\right)$$

Now,

$$v=r\Omega$$

$$=-\frac{c\,r}{2\nu}\log\frac{r}{a}+\frac{c\,b^2}{2\nu r}\left(\frac{r^2-a^2}{b^2-a^2}\right)\log\frac{b}{a}$$

$$=\frac{1}{2}\frac{c}{\nu}\left[\frac{b^2}{r}\left(\frac{r^2-a^2}{b^2-a^2}\right)\log\frac{b}{a}-r\log\frac{r}{a}\right]$$

#

Ex. Prove that in the steady motion of a viscous liquid in the two-dimensionst

$$\nu \nabla^4 \psi = \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}$$

where (X, Y) is the impressed force per unit area.

Sol^n. The Navier-Stokes equation of motion

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\vec{\nabla})\vec{q} = \vec{F} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{q} \longrightarrow (i)$$

For steady motion (i) takes the form

$$(\vec{q}\cdot\vec{\nabla})\vec{q} = \vec{F} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{q} \longrightarrow (ii)$$

for slow motion, $(\vec{q}\cdot\vec{\nabla})\vec{q} = 0$

$\therefore$ (ii) $\Rightarrow$ $\vec{F} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{q} = 0$

Taking curl on both sides.

$$\text{curl}\,\vec{F} - \text{curl}\left(\frac{1}{\rho}\nabla p\right) + \text{curl}\left(\nu \nabla^2 \vec{q}\right) = 0$$

$\Rightarrow \text{curl}\,\vec{F} + \nu \nabla^2 \vec{\zeta} = 0 \longrightarrow$ (iii), where $\vec{\zeta} = \text{curl}\,\vec{q}$

Now, the stream function ψ exists, then $u = -\frac{\partial \psi}{\partial y}$, $v = \frac{\partial \psi}{\partial x}$

$$\therefore \vec{\zeta} = \text{curl}\,\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & 0 \end{vmatrix}$$

$$= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$= \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$$

$$= \nabla^2 \psi$$

$\therefore$ (iii) $\Rightarrow$ $\text{curl}\,\vec{F} + \nu \nabla^2(\nabla^2\psi) = 0$

$\Rightarrow \text{curl}\,\vec{F} + \nu \nabla^4\psi = 0$

$\Rightarrow \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) + \nu \nabla^4\psi = 0$

$\Rightarrow \nu \nabla^4\psi = \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}$

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Ex. The space between two coaxial cylinder of radii a and b is filled with viscous fluid and the cylinders are made to rotate with angular velocities Ω_1 and Ω_2. Prove that in steady motion, the angular velocity of the fluid is given by

$$\Omega = \frac{(b^2 - r^2)\Omega_1 - b^2(r^2 - a^2)\Omega_2}{r^2(b^2 - a^2)}.$$