Q.3 Solve : $(1-x^2)y'' - xy' + 4y = 0$ in Series

OR [UPTU, U.P – 2014, 2012, 2006]

Note Solve in series Chebyshev's Differential Eq. (when $n=2$)

OR

Solve the Differential Equation $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$

about the point $x=0$.

Note?

OR

Solve the Differential Equation $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$

Note near the point $x=0$.

OR

Note

Find Power series Solution of the Differential Equation

$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$ about the ordinary Point $x=0$. Note?

Rough work

① $$\sum_{n=2}^{\infty} x^{n-2} = x^0 + x^1 + x^2 + x^3 \text{———} \infty$$

$$\sum_{n=1} x^{n-1}$$

$$\sum_{n=0}^{\infty} x^n = x^0 + x^1 + x^2 + x^3 \text{———} \infty$$

Same.

② $$\sum_{n=0}^{\infty} x^{n+1} = x^1 + x^2 + x^3 \text{———} \infty$$

$$\sum_{n=1}^{\infty} x^n = x^1 + x^2 + x^3 \text{———} \infty$$

Same

Indexing:- $$\sum_{\downarrow\uparrow} (\uparrow\downarrow)$$

Sol:- The given Differential Equation is :-

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0 \quad \text{——} ①$$

$\boxed{\text{Chebyshev's D.E when } n = 2}$

which is the required Homogeneous Second order Linear Diff. Eq. with variable coefficients.

Here :-

$P_0(x) = 1-x^2 \quad , \quad P_1(x) = -x \quad , \quad P_2(x) = 4$

At Point $x=0$:- $P_0(0) = 1-(0)^2 = 1 \neq 0$

By Default

$\Downarrow$

$x = 0$ is an Ordinary Point

$\Downarrow$

Power Series method.

Now by Power Series Method :-

Let solution is $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ —②

(18)

Now $\frac{dy}{dx} = \sum_{n=1}^{\infty} a_n (n x^{n-1}) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} (n a_n)(n-1) x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Now Put y, $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in given D.E (1), we get:-

$$\Rightarrow (1-x^2)\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) - x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2+2} - \sum_{n=1}^{\infty} n a_n x^{n-1+1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

[By Indexing]

Same ?

Note?

$$\Rightarrow \left[(2)(1) a_2 x^0 + (3)(2) a_3 x^1 + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n\right] - \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

$$- \left[1 a_1 x^1 + \sum_{n=2}^{\infty} n a_n x^n\right] + 4\left[a_0 x^0 + a_1 x^1 + \sum_{n=2}^{\infty} a_n x^n\right] = 0$$

$$\Rightarrow 2a_2 + 6a_3 x - a_1 x + 4a_0 + 4a_1 x + \sum_{n=2}^{\infty} \left[\begin{array}{l}(n+2)(n+1) a_{n+2} - n(n-1) a_n \\ - n a_n + 4a_n\end{array}\right] x^n = 0$$

$$\Rightarrow (4a_0 + 2a_2) + (6a_3 + 3a_1)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n^2-n)a_n - na_n + 4a_n \right] x^n = 0$$

$$\Rightarrow (4a_0 + 2a_2) + (6a_3 + 3a_1)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + (-n^2 + \cancel{n} - \cancel{n} + 4)a_n \right] x^n = 0$$

$$\Rightarrow (4a_0 + 2a_2) + (6a_3 + 3a_1)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + (4-n^2)a_n \right] x^n = 0$$

By comparing Coff. of Powers of x to zero.

Now coff of $x^0 \Rightarrow 4a_0 + 2a_2 = 0 \Rightarrow 2a_2 = -4a_0 \Rightarrow \boxed{a_2 = -2a_0}$

Coff. of $x^1 \Rightarrow 6a_3 + 3a_1 = 0 \Rightarrow 6a_3 = -3a_1 \Rightarrow \boxed{a_3 = \frac{-a_1}{2}}$

Coffi. of $x^n \Rightarrow (n+2)(n+1)a_{n+2} + (4-n^2)a_n = 0 \quad \forall\ n \geq 2$

$$\Rightarrow (n+2)(n+1)a_{n+2} = -(4-n^2)a_n$$

$$\Rightarrow \quad a_{n+2} = \frac{(n^2-4)\,a_n}{(n+2)(n+1)}$$

$$\Rightarrow \quad a_{n+2} = \frac{\cancel{(n+2)}(n-2)\,a_n}{\cancel{(n+2)}(n-1)}$$

$$\Rightarrow \quad \boxed{a_{n+2} = \frac{(n-2)}{(n-1)}\,a_n} \quad \text{—} (2) \quad \forall\ n \geq 2$$

Note

it is called as Recurrance Relation or Difference Eq.

Putt $n=2$ in eq. (2) :- $a_4 = \frac{(0)}{(1)}\,a_2 \Rightarrow \boxed{a_4 = 0}$

Put $n=3$ in eq. (2) :- $a_5 = \frac{(1)}{2}\,a_3 = \frac{1}{2}\left(-\frac{1}{2}a_1\right) = \boxed{-\frac{1}{4}\,a_1 = a_5}$

Put $n=4$ in eq. (2) :- $a_6 = \frac{(2)}{(3)}\,a_4 \Rightarrow \boxed{a_6 = 0}$

Now put the values of a_2, a_3, a_4, a_5 & a_6 in solution:-

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \text{———} \infty$$

$$= a_0 + a_1 x + (-2a_0) x^2 + \left(-\frac{a_1}{2}\right) x^3 + (0) x^4 + \left(-\frac{1}{4} a_1\right) x^5 + (0) x^6 + \text{———} \infty$$

$$= a_0 + a_1 x - 2a_0 x^2 - \frac{a_1}{2} x^3 - \frac{1}{4} a_1 x^5 + \text{———} \infty$$

$$\boxed{y = a_0\left(1 - 2x^2 + \text{———} \infty\right) + a_1\left(x - \frac{1}{2} x^3 - \frac{1}{4} x^5 + \text{———} \infty\right)}$$

which is the required solution of given D.E (B) &

Q.4. Solve in series the Legendre's Diff. Eq. (when $n=p$)

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0$$

OR

Solve : $(1-x^2)y'' - 2xy' + p(p+1)y = 0$ in series.

OR

Solve : $(1-x^2)y_2 - 2xy_1 + p(p+1)y = 0$ in series.

OR

Solve the Diffr. Eq. $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0$

Note ? → about the point $x=0$

OR

Solve the Diffi. Eq. $(1-x^2)y'' - 2xy' + p(p+1)y = 0$

Note → near the point $x=0$.

OR

Find Power Series Solution of the Differential Equation

$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0$ about the Ordinary Point $x=0$.

OR

Solve the Diffi. Eq. $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0$ in the powers of x.

Note? $\boxed{a=0}$

SPPU, Pune – 2018, 2017
HPTU, Himachal – 2017, 2016
UPTU, U.P – 2010
RTU, Rajasthan – 2002

25

Sol:- The given Legendre's Differential Equation (with $n=p$) is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0 \quad \text{—} (1)$$

which is the required Homogeneous Second Order Linear Differential Equation with Variable Coefficients.

Here:-

$$P_0(x) = 1-x^2 \quad ; \quad P_1(x) = -2x \quad ; P_2(x) = p(p+1)$$

Now at Point $x=0$:- $P_0(0) = 1-(0)^2 = 1 \neq 0$

$\downarrow$

Ordinary Point

$\downarrow$

Power Series Method.

Now by Power Series Method:-

Let the solution be $y = \sum_{n=0}^{\infty} a_n x^n$

Now $\dfrac{dy}{dx} = \sum_{\boxed{n=1}}^{\infty} n a_n x^{n-1}$, $\dfrac{d^2y}{dx^2} = \sum_{\boxed{n=2}}^{\infty} n(n-1) a_n x^{n-2}$

Note

Now put the values of y, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in eq. ① , we get:-

$$(1-x^2)\left[\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right] - 2x\left[\sum_{n=1}^{\infty} n a_n x^{n-1}\right] + p(p+1)\left[\sum_{n=0}^{\infty} a_n x^n\right] = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2+2} - 2\sum_{n=1}^{\infty} n a_n x^{n-1+1} + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

Indexing

Same ?

$$\Rightarrow \left[(2)(1) a_2 x^0 + (3)(2) a_3 x^1 + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n\right] - \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

$$-2\left[1 a_1 x^1 + \sum_{n=2}^{\infty} n a_n x^n\right] + p(p+1)\left[a_0 x^0 + a_1 x^1 + \sum_{n=2}^{\infty} a_n x^n\right] = 0$$

$$\Rightarrow \left[2a_2 + 6a_3 x - 2a_1 x + p(p+1) a_0 + p(p+1) a_1 x\right] +$$

$$\sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} - n(n-1) a_n - 2n a_n + p(p+1) a_n\right] x^n = 0$$

$$\Rightarrow \left[2a_2 + p(p+1)a_0\right] + \left[6a_3 - 2a_1 + p(p+1)a_1\right]x +$$

$$\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + \left[-n^2+n-2n+p(p+1)\right]a_n\right]x^n = 0$$

$$\Rightarrow \left[2a_2 + p(p+1)a_0\right] + \left[6a_3 + (p^2+p-2)a_1\right]x +$$

$$\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + (-n^2-n+p^2+p)a_n\right]x^n = 0$$

$$\Rightarrow \left[2a_2 + p(p+1)a_0\right] + \left[6a_3 + (p+2)(p-1)a_1\right]x +$$

$$\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} - \left[n^2-p^2+\underline{n-p}\right]a_n\right]x^n = 0$$

$$\Rightarrow \left[2a_2 + p(p+1)a_0\right] + \left[6a_3 + (p+2)(p-1)a_1\right]x +$$

$$\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} - (n-p)(\underline{n+p}+1)a_n\right]x^n = 0$$

Scanned with CamScanner

Now By Equating coffi of various powers of $x = 0$

Coffi. of x^0 :- $2a_2 + p(p+1)a_0 = 0 \Rightarrow \boxed{a_2 = \frac{-p(p+1)}{2} a_0}$

Coffi of x^1 :- $6a_3 + (p+2)(p-1)a_1 = 0 \Rightarrow \boxed{a_3 = -\frac{(p+2)(p-1)}{6} a_1}$

Coeffi. of x^n :- $(n+2)(n+1)a_{n+2} - (n-p)(n+p+1)a_n = 0 \quad \forall\ n \geq 2$

$$\Rightarrow (n+2)(n+1)a_{n+2} = (n-p)(n+p+1)a_n$$

$$\boxed{a_{n+2} = \frac{(n-p)(n+p+1)}{(n+2)(n+1)} a_n} \quad \forall\ n \geq 2 \quad -(2)$$

which is the required Recurrance Relation or Difference Eq.

Now Put $n = 2$ in eq. (2) :- $a_4 = \frac{(2-p)(3+p)}{(4)(3)} \cdot a_2$

$$a_4 = -\frac{(p-2)(p+3)}{12}\left[-\frac{p(p+1)}{2}a_0\right]$$

$$\boxed{a_4 = \frac{(p-2)\,p\,(p+1)(p+3)}{24}a_0}$$

Now put $n=3$ in eq. ②:- $\quad a_5 = \frac{(3-p)(p+4)}{(5)(4)}a_3$

$$= -\frac{(p-3)(p+4)}{20}\left[-\frac{(p+2)(p-1)}{6}a_1\right]$$

$$\boxed{a_5 = \frac{(p-3)(p-1)(p+2)(p+4)}{120}a_1}$$

Now put the values of a_2, a_3, a_4 & a_5 in solution :-

Scanned with CamScanner

(31)

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \text{---} \infty$$

$$= a_0 + a_1 x + \left[\frac{-p(p+1)}{2} a_0 \right] x^2 + \left[\frac{-(p+2)(p-1)}{6} a_1 \right] x^3$$

$$+ \left[\frac{(p-2)\, p\,(p+1)(p+3)}{24} a_0 \right] x^4 + \left[\frac{(p-3)(p-1)(p+2)(p+4)}{120} a_1 \right] x^5$$

$$+ \text{---} \infty$$

$$\boxed{\begin{aligned} y &= a_0 \left[1 - \frac{p(p+1)}{2} x^2 + \frac{(p-2)\, p\,(p+1)(p+3)}{24} x^4 + \text{---} \infty \right] \\ &+ a_1 \left[x - \frac{(p-1)(p+2)}{6} x^3 + \frac{(p-3)(p-1)(p+2)(p+4)}{120} x^5 - \infty \right] \end{aligned}}$$

which is the required solution of given D.E (1)