Assumptions for non-static cosmological models:

(a) There exists a cosmic time t which is orthogonal to the spatial geometry so that in spherical polar coordinates the line element may be taken as

$$ds^2 = dt^2 - e^{\mu(r,t)}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi)$$

where the coordinates r and t are so choosen that the spherical surface r = constant moves with the meterial lying on its surface. Such a coordinate system is called comoving co-ordinate system.

(b) The three dimensional spatial surfaces belonging to t = constant are locally isotropic and homogeneous, i.e. at any epoch the universe is same everywhere in space and in every direction so that the function $\mu(r,t)$ may be taken as $\mu(r,t) = f(r) + g(t)$.

(*) Derivation of Friedmann - Robertson - Walker (FRW) line-element:

The non-static spherically symmetric line-element in comoving co-ordinates is given by

$$ds^2 = dt^2 - e^{\mu}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2), \quad (1)$$

where $\mu = \mu(r,t)$ is a function of r and t only.

Due to the hypothesis of isotropy and homogeneity of the 3-space, the function $\mu(r, t)$ must be a function of the form

$$\mu(r,t) = f(r) + g(t) \longrightarrow (2)$$

where the function $f(r)$ and $g(t)$ are to be determined.

Let $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, $x^4 = t$.

Then the line element (1) takes the form

$$ds^2 = g_{ij}\, dx^i dx^j$$

where

$$g_{11} = -e^{\mu}, \quad g_{22} = -e^{\mu} r^2$$

$$g_{33} = -e^{\mu} r^2 \sin^2\theta, \quad g_{44} = 1$$

and $g_{ij} = 0$ for $i \neq j$.

Therefore the christoffel symbols of second kind are given by

$$\Gamma^i_{ii} = \frac{1}{2g_{ii}} \frac{\partial g_{ii}}{\partial x^i} \longrightarrow (3)$$

$$\Gamma^i_{ij} = \frac{1}{2g_{ii}} \frac{\partial g_{ii}}{\partial x^j} \longrightarrow (4)$$

$$\Gamma^k_{ii} = -\frac{1}{2g_{kk}} \frac{\partial g_{ii}}{\partial x^k}, \quad i \neq k \longrightarrow (5)$$

$$\Gamma^k_{ij} = 0, \quad i \neq j \neq k \longrightarrow (6)$$

From (3), we have

$$\Gamma^1_{11} = \frac{1}{2g_{11}} \frac{\partial g_{11}}{\partial x^1} = \frac{1}{2(-e^{\mu})} \frac{\partial}{\partial r}(-e^{\mu})$$

$$= \frac{1}{-2e^{\mu}} (-e^{\mu}) \frac{\partial \mu}{\partial r}$$

$$= \frac{1}{2}\mu' \quad \text{where } \mu' = \frac{\partial \mu}{\partial r}$$

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$\Rightarrow \Gamma^1_{11} = \frac{1}{2} f'$ $\quad \because \mu(r,t) = f(r) + g(t)$.

From (4) we have

$$\Gamma^2_{21} = \frac{1}{2g_{22}} \frac{\partial g_{22}}{\partial x^1}$$

$$= \frac{1}{-2e^{\mu} r^2} \frac{\partial}{\partial r}(-e^{\mu} r^2)$$

$$= \frac{1}{2e^{\mu} r^2}(e^{\mu}.2r + \mu' e^{\mu} r^2)$$

$$= \frac{1}{r} + \frac{1}{2} f'$$

$$\Gamma^3_{31} = \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial x^1}$$

$$= \frac{1}{-2e^{\mu} r^2 \sin^2\theta} \frac{\partial}{\partial r}(-e^{\mu} r^2 \sin^2\theta)$$

$$= \frac{1}{2e^{\mu} r^2 \sin^2\theta} \quad e^{\mu}.2r + \mu' e^{\mu} r^2 \sin^2\theta$$

$$= \frac{1}{r} + \frac{1}{2} f'$$

$$\Gamma^3_{32} = \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial x^2}$$

$$= \frac{1}{-2e^{\mu} r^2 \sin^2\theta} \frac{\partial}{\partial \theta}(-e^{\mu} r^2 \sin^2\theta)$$

$$= \frac{1}{2e^{\mu} r^2 \sin^2\theta} \quad e^{\mu} r^2 . 2\sin\theta\cos\theta$$

$$= \cot\theta .$$

$$\Gamma^1_{14} = \frac{1}{2g_{11}} \frac{\partial g_{11}}{\partial x^4}$$

$$= \frac{1}{2(-e^{\mu})} \frac{\partial}{\partial t}(-e^{\mu})$$

$$= \frac{1}{2e^{\mu}} e^{\mu} \frac{\partial \mu}{\partial t}$$

$$= \frac{\dot{\mu}}{2} \quad \text{where } \dot{\mu} = \frac{\partial \mu}{\partial t}$$

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$$\Rightarrow \Gamma^1_{14} = \frac{1}{2}\dot{g} \quad ; \quad \mu(r,t) = f(r) + g(t)$$

$$\Gamma^2_{24} = \frac{1}{2g_{22}} \frac{\partial g_{22}}{\partial x^4}$$

$$= \frac{1}{2(-e^{\mu} r^2)} \frac{\partial}{\partial t}(-e^{\mu} r^2)$$

$$= \frac{1}{2e^{\mu} r^2}\left(e^{\mu} \frac{\partial \mu}{\partial t}\right) r^2$$

$$= \frac{1}{2}\dot{g}$$

$$\Gamma^3_{34} = \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial x^4} = \frac{1}{2(-e^{\mu} r^2 \sin^2\theta)} \frac{\partial}{\partial t}(-e^{\mu} r^2 \sin^2\theta)$$

$$= \frac{1}{2e^{\mu} r^2 \sin^2\theta}\left(e^{\mu} \frac{\partial \mu}{\partial t}\right) r^2 \sin^2\theta$$

$$= \frac{1}{2}\dot{g}$$

From (5), we have

$$\Gamma^4_{11} = -\frac{1}{2g_{44}} \frac{\partial g_{11}}{\partial x^4} = -\frac{1}{2}\frac{\partial}{\partial t}(-e^{\mu}) = \frac{1}{2}\dot{g}e^{\mu}$$

$$\Gamma^1_{22} = -\frac{1}{2g_{11}} \frac{\partial g_{22}}{\partial x^1} = -\frac{1}{2(-e^{\mu})} \frac{\partial}{\partial r}(-e^{\mu} r^2)$$

$$= -\frac{1}{2e^{\mu}}\left(e^{\mu}.2r + e^{\mu}\mu' r^2\right)$$

$$= -\left(r + \frac{1}{2} f'(r^2)\right)$$

$$\Gamma^4_{22} = -\frac{1}{2g_{44}} \frac{\partial g_{22}}{\partial x^4}$$

$$= -\frac{1}{2}\frac{\partial}{\partial t}(-e^{\mu} r^2)$$

$$= \frac{1}{2}\dot{g}e^{\mu} r^2$$