Statistical Mechanics
Lecture 9

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Fermi-Dirac Statistics:

Fermi-Dirac statistics is a type of quantum statistics that applies to a system consisting of many identical particles that obey the Pauli's Exclusion Principle. It is applies to identical and indistinguishable particles with half integer spin. For the case of negligible interaction between the particles, the system can be described in case of single particle energy state. As a result the Fermi-Dirac distribution of particles over these states where no two particles can occupy the same states, which have a considerable effect on the system. The particles known as fermions which have spin $^1/_2$.

Let $N_1, N_2, N_3, \dots \dots N_n$ are a system of N indistinguiable particles with energy $E_1, E_2, E_3, \dots \dots E_n$ and g_i is the multiplicity or degeneracy of energy level. The number of ways in which N_i particles are arranged in g_i quntum state is given by

$$ {}^{g_i}_{N_i}c = \frac{g_i!}{N_i!\,(g_i - N_i)!} \rightarrow (i) $$

This equation results that the particles are indistinguable and each quantum state can accommodate only one particle in accordance with Pauli's Exclusion Principle.

The total number of ways W of distributing $N_1, N_2, \dots . N_n$ particles in N energy level in the product of equation (i)

over all level.

$$N = \prod_{i=1}^{n} \frac{g_i!}{N_i!\,(g_i - N_i)!} \to (ii)$$

Where $\prod \to Product, g_i \gg 1, N_i \gg i, (g_i - N_i) \gg 1$

Applying Sterling Theorem we get

$$lnW = \sum_{i=1}^{n} [g_i(lng_i - 1) - N_i(lnN_i - 1) - (g_i - N_i)\{ln(g_i - N_i) - 1\}]$$

$$lnW = \sum_{i=1}^{n} [g_i ln g_i - N_i ln N_i - (g_i - N_i) \, ln(g_i - N_i)] \rightarrow (iii)$$

Since the system is in equilibrium to find most probable distribution W or lnW must be maximised subject to restrictions that the total number of particle N and total energy must be constant i.e.

$$N = \sum_{i=1}^{n} N_i = Contant \rightarrow (iv)$$

$$U = \sum_{i=1}^{n} N_i E_i = Constant \rightarrow (v)$$

Applying Lagrange's method from equation $(iii), (iv)$ and (v) we get

$$\sum_{i=0}^{i} [ln(g_i - N_i) - lnN_i + \alpha + \beta E_i]\delta N_i = 0$$

$$\Longrightarrow ln(g_i - N_i) - lnN_i + \alpha + \beta E_i = 0 \rightarrow (vi)$$

As $\alpha\ and\ \beta$ are arbitrarily constant, this equation gives

$$f(E_i) = \frac{N_i}{g_i} = \frac{1}{1 + e^{-\alpha-\beta E_i}} \rightarrow (vii)$$

Which is the Fermi-Dirac Distribution Function. Here

$$\alpha = -\frac{E_F}{K_\beta T}$$

$$\beta = -\frac{1}{K_B T}$$

Here $K_B \rightarrow Boltzmann\ Constant\ and\ T \rightarrow Absolute\ Temparature$

Therefore Fermi-Dirac Distribution Function takes the form as

$$f(E_i) = \frac{1}{1+e^{(E_i - E_g)/K_\beta T}} \rightarrow (viii)$$

Since only one particle may occupy a quantum state, $f(E_i)$ for Fermi-Dirac statistics is the probability that a quantum state of energy E_i is occupied.

At

$$T = 0, f(E_i) = 1\ for\ E_i > E_F$$

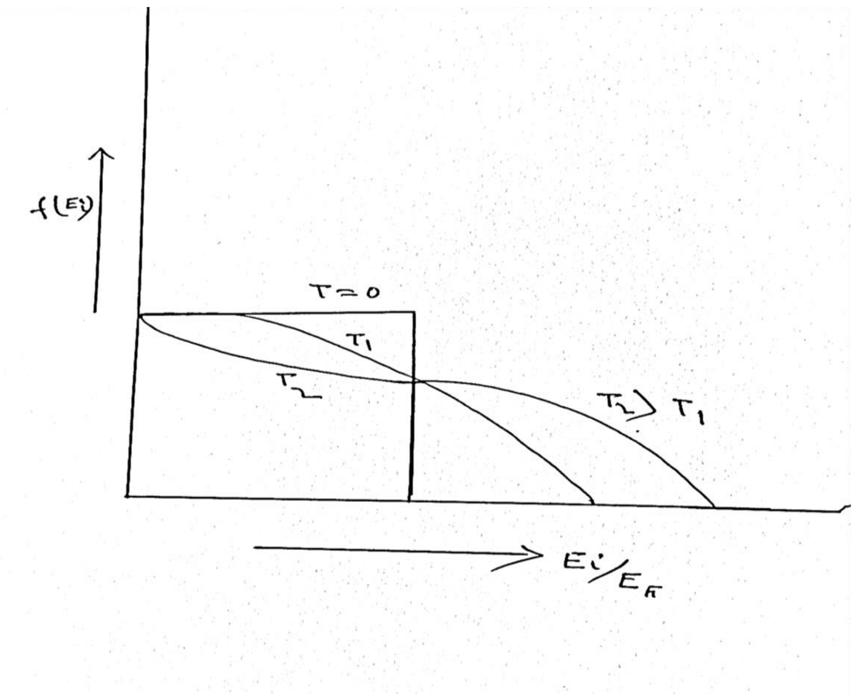
And

$$f(E_i) = 0\ \ for\ E_i < E_F$$

Thus at absolute zero of temperature, $f(E_i)$ is a step function. The probability of occupation of all states with energy less than E_F is zero. Thus at absolute zero of temperature, the Fermi level represent the highest occupied energy state.

At $T > 0, f(E_i)$ is close to unity for $E_i \ll E_F$ and approaches zero for $E_i \gg E_F$. For variation of $f(E_i)$ for two different representation $T_1\ and\ T_2$, if temperature is not very large $f(E_i)$ varies rapidly from about unity to almost zero over an energy range of few times of $K_\beta T$ around E_F.

At a nonzero temperature equation $(viii)$ shows that $f(E_i) = \frac{1}{2}$ at $E_F = E_i$. Thus Fermi level is that energy level for which the probability of occupation at $T > 0$ is $\frac{1}{2}$.



At low temperature when $f(E_i)$ is the nearly a step function the distribution function is said to be strongly degenerated. At every high temperature when step like character is lost it is said to be nearly nongenerated.

Another Method:

In this case of Fermi-Dirac statistics the Partition Function is

$$Z = \left[1 + e^{\beta(\mu - \epsilon_1)}\right]\left[1 + e^{\beta(\mu - \epsilon_2)}\right] \ldots \ldots .. \rightarrow (i)$$

Therefore

$$lnZ = \sum_s ln\left[1 + e^{\beta(\mu - \epsilon_s)}\right]$$

Hence

$$N = \frac{1}{\beta} \frac{\partial lnZ}{\partial \mu}$$

$$N = \sum_{s} \frac{e^{\beta(\mu - \epsilon_s)}}{1 + e^{\beta(N - \epsilon_s)}}$$

$$N = \sum < n_s > \rightarrow (ii)$$

Therefore

$$< n_s > = \frac{1}{e^{\beta(\epsilon_s - \mu_s)}}$$

This shows that if ϵ_s is very large as $n_s \rightarrow 0$. On the other hand if ϵ_s is small, since denominator is always greater than one we have

$$< n_s > \leq 1$$

The behaviour of the gas obeying Fermi-Dirac statistics is different from that obeying Bose-Einstein statistics. This difference becomes particularly striking as tends to zero when the gas is in state of lowest energy. In the case of Bose-Enstein statistics since there is no restriction on the number of particles to be placed in any single particle state even though the gas has lowest energy, one is forced to populate the successive states of higher energy with one particle in each.

This lowest energy of a gas obeying Fermi-Dirac statistics therefore is much higher than that it would have been if the particles has obeyed Fermi-Dirac statistics.

Fermi Energy and Momentum:

The Fermi-Dirac distribution function which gives the avarge occupation of energy level ϵ is given by

$$f(\epsilon) =< n_\epsilon > = \frac{1}{e^{\epsilon-\mu/KT} + 1} \rightarrow (i)$$

Let $T = O$, then the value of μ is μ_F .

Now at $T = 0$, $< n_\epsilon >$ has two possible values

$$< n_\epsilon > = \frac{1}{e^{-\alpha} + 1} = 1\,, \qquad if\ \epsilon < \mu_F \ \rightarrow (ii)$$

$$< n_\epsilon > = \frac{1}{e^{\alpha} + 1} = 0, \qquad if\ \epsilon > \mu_F \ \rightarrow (iii)$$

At absolute zero pf temperature, the fermions will necessarily occupy the lowest available energy state. An imidiate consequences of the Pauli's Exclusion Principle is that each quantum state can contain only one fermion.

Therefore all the lowest states will be occupied with one fermion in each, until all the fermions are accommodated under this condition the gas is to be degenerated.

$$< n_\epsilon > = \frac{1}{e^{\epsilon-\mu/KT}+1} = \frac{1}{2} - \frac{(\epsilon-\mu)}{4KT} + \ \ldots . \rightarrow (vi)$$

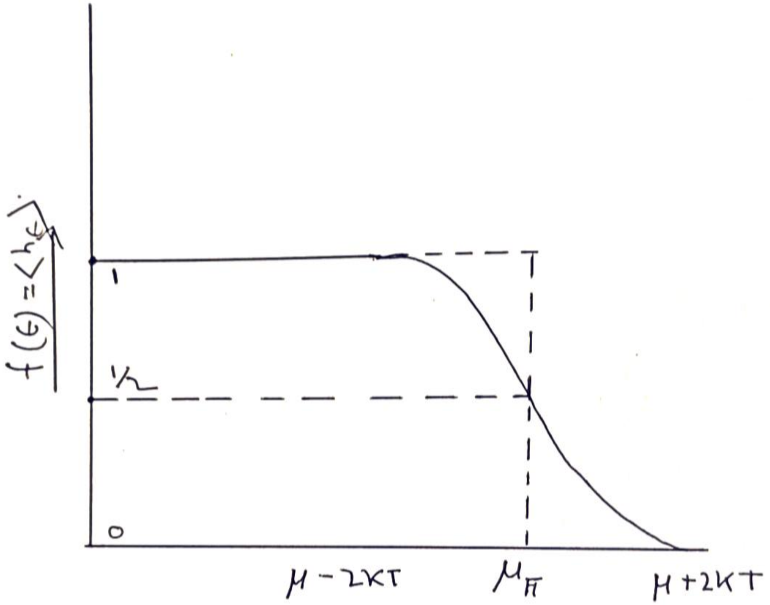
Therefore if

$$If\ \epsilon \leq \mu - 2KT, \qquad < n_\epsilon > = 1 \rightarrow (v)$$

$$If\ \epsilon \geq \mu + 2KT, \qquad < n_\epsilon > = 0 \ \rightarrow (vi)$$

$$If\ \epsilon = \mu, \qquad < n_\epsilon > = \frac{1}{2} \rightarrow (vii)$$

This can be shown graphically as



The region of ϵ when $< n_\epsilon >$ changes from unity to zero $(from\ \mu - 2KT\ to\ \mu + 2KT)$ narrows as the temperature decreases and absolute zero becomes a sharp discontinuity. The distribution takes the form of a right angle. In the figure at $T = 0$ all states with $\epsilon < \mu_F$ are occupied, while those with energy $\epsilon > \mu_F$ are empty. The highest occupied level is called Fermy level and is repeated by ϵ_F . Below this level there are exactly N states where N is the total number of Fermions. The states above this energy level are unoccupied. The value of ϵ_F depends on the numbers of Fermions at $T = 0,\ \ \epsilon_F\ coincides\ with\ \mu_F$. The value of μ_F as

$$\int_0^\infty n(\epsilon)\, d\epsilon = N \rightarrow (viii)$$

At $T = 0$ all single particle states up to ϵ_F are completely filled with one particle in per state .

$For\ \epsilon < \epsilon_F$, the number of states $\sigma(\epsilon) = n(\epsilon) \rightarrow (ix)$

$For\ \epsilon > \epsilon_F$, the number of states $\sigma(\epsilon) = 0\ \rightarrow (x)$

Therefore

$$\int_0^{\epsilon_F} \sigma(\epsilon)\, d\epsilon = N \rightarrow (xi)$$

If the particles have spin s then there are $(2s + 1)$ single particle states, all having the same energy ϵ . Therefore all density of state $\sigma(\epsilon)d\epsilon$ in the energy range from $\epsilon\ to\ \epsilon + d\epsilon$ is given by

$$\sigma(\epsilon)d\epsilon = \frac{V2m^{3/2}(2s+1)}{4\pi^2h^3}d\epsilon \rightarrow (xii)$$

Hence equation (xi) can be written as

$$\int_0^{\epsilon_F} \frac{V(2s+1)(2m)^{3/2}}{4\pi^2h^3}\ \epsilon^{1/2}d\epsilon = N \rightarrow (xiii)$$

$$\Longrightarrow \frac{(2s+1)V}{4\pi^2h^3}\ (2m)^{3/2}\ \frac{2}{3}\ \epsilon_F^{3/2} = N$$

$$\Longrightarrow \epsilon_F = \frac{h^2}{2m}\left(\frac{6N\pi^2}{(2s+1)V}\right)^{2/3} \rightarrow (xiv)$$

Which is the expression of Fermi Energy. The value of single particle momentum corresponding to Fermi Energy is referred to a Fermi Momentum and denoted by

$$P_F = \sqrt{2m\epsilon_F} \rightarrow (xv)$$

$$P_F = h\left[\frac{6N\pi^2}{(2s+1)V}\right]^{2/3} \rightarrow (xvi)$$