Diffusion of vorticity from a line:

We consider the flow of an incompressible fluid through a vortex filament of strength K along the axis of z in an infinite mass of fluid. Motion of the fluid particles will be in circles about the z-axis and velocity at a distance r of the vortex will be function of r and t.

Here vorticity components along r and θ are zero and the velocity component along z-direction is ζ (say)

Equation of vorticity $\vec{\xi}$ is

$$\frac{D\vec{\xi}}{Dt} = (\vec{\xi}\cdot\vec{\nabla})\vec{q} + \nu\nabla^2\vec{\xi} \longrightarrow (i)$$

Since $\vec{\xi} = \hat{i}.0 + \hat{j}.0 + \hat{k}.\zeta$

Equation (i) can be written as

$$\frac{D\zeta}{Dt} = \zeta\frac{\partial}{\partial z}(\hat{i}u + \hat{j}v) + \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\zeta$$

$$= \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\zeta$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\zeta = \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\zeta$$

$$\Rightarrow \frac{\partial\zeta}{\partial t} + u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y} = \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\zeta \longrightarrow (ii)$$

Since, velocity is a function of r and t, therefore we take $u = -\frac{y}{r}f(r,t)$, $v = \frac{x}{r}f(r,t)$, where $r = \sqrt{x^2+y^2}$

This form of velocity component is satisfy the continuity equation i.e.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -y \frac{\partial}{\partial r}\left(\frac{f(r,t)}{r}\right)\frac{\partial r}{\partial x} + x\left(\frac{\partial}{\partial r}\left(\frac{f(r,t)}{r}\right)\right)\frac{\partial r}{\partial y}$$

$$= -y \frac{\partial}{\partial r}\left[\frac{f(r,t)}{r}\right]\frac{2x}{2r} + x\frac{\partial}{\partial r}\left[\frac{f(r,t)}{r}\right]\frac{y}{r}$$

$$= 0$$

Now, $$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$= \frac{\partial}{\partial x}\left[\frac{x}{r} f(r,t)\right] + \frac{\partial}{\partial y}\left[\frac{y}{r} f(r,t)\right]$$

$$= \frac{1}{r} f(r,t) + x\frac{\partial}{\partial r}\left[\frac{1}{r} f(r,t)\right]\frac{\partial r}{\partial x} + \frac{1}{r} f(r,t) + y\frac{\partial}{\partial r}\left(\frac{f}{r}\right)\frac{\partial r}{\partial y}$$

$$= \frac{2}{r} f(r,t) + x\frac{\partial}{\partial r}\left(\frac{f}{r}\right)\frac{x}{r} + y\frac{\partial}{\partial r}\left(\frac{f}{r}\right)\frac{y}{r}$$

$$= \frac{2}{r} f(r,t) + \frac{\partial}{\partial r}\left(\frac{f}{r}\right)\left(\frac{x^2}{r} + \frac{y^2}{r}\right)$$

$$= \frac{2}{r} f(r,t) + r\frac{\partial}{\partial r}\left[\frac{f(r,t)}{r}\right]$$

$$= \phi(r,t)$$

Again, $$\frac{\partial \zeta}{\partial x} = \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial x} = \frac{x}{r}\frac{\partial \phi}{\partial r}$$

$$\frac{\partial \zeta}{\partial y} = \frac{\partial \phi}{\partial r}\cdot\frac{\partial r}{\partial y} = \frac{y}{r}\frac{\partial \phi}{\partial r}$$

L.H.S. of (ii)

$$u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}=u\frac{x}{r}\frac{\partial\phi}{\partial r}+v\frac{y}{r}\frac{\partial\phi}{\partial r}$$

$$=-\frac{y}{r}f(r,t)\frac{x}{r}\frac{\partial\phi}{\partial r}+\frac{x}{r}f(r,t)\frac{y}{r}\frac{\partial\phi}{\partial r}$$

$$=0$$

From equation (ii) can be written as

$$\frac{\partial\phi}{\partial t}=\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi(r,t)$$

$$=\nu\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{r^2\partial\theta^2}\right]\phi(r,t)$$

$$=\nu\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right]\phi(r,t) \qquad ;\ \phi \text{ is not contain } \theta .$$

$$\Rightarrow \frac{\partial\phi}{\partial t}=\nu\left[\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}\right]$$

Solution is $\phi=\frac{k}{4\pi\nu t}e^{-\frac{r^2}{4\nu t}}$

Total circulation in a circle of radius 'r' is given by

$$\int_{\theta=0}^{2\pi}\int_0^r \phi\, r\,dr\,d\theta \qquad ;\ K = \text{total amount of viscosity}$$

$$=2\pi\int_0^r \frac{k}{4\pi\nu t}e^{-\frac{r^2}{4\nu t}}\,r\,dr$$

$$=K\left[1-e^{-\frac{r^2}{4\nu t}}\right]$$

Now, $\phi(r,t) = \frac{2}{r} f(r,t) + r \frac{\partial}{\partial r}\left[\frac{f(r,t)}{r}\right]$

$$= \frac{k}{4\pi\nu t} e^{-\frac{r^2}{4\nu t}}$$

$$\Rightarrow \frac{2f}{r} + r\left[\frac{1}{r}\frac{\partial f}{\partial r} - \frac{f}{r^2}\right] = \frac{k}{4\pi\nu t} e^{-\frac{r^2}{4\nu t}}$$

$$\Rightarrow \frac{\partial f}{\partial r} + \frac{f}{r} = \frac{k}{4\pi\nu t} e^{-\frac{r^2}{4\nu t}} \quad ;$$

$$\Rightarrow \frac{\partial}{\partial r}(rf) = \frac{kr}{4\pi\nu t} e^{-\frac{r^2}{4\nu t}}$$

$r^2 = z$

$2r\,dr = dz$

$$\int \frac{kr}{4\pi\nu t} e^{-\frac{r^2}{4\nu t}} dr$$

$$= \frac{k}{4\pi\nu t}\int \frac{1}{2} dz\, e^{-\frac{z}{4\nu t}}$$

$$= \frac{k}{8\pi\nu t} e^{-\frac{z}{4\nu t}}\left(\frac{4\nu t}{1}\right)$$

$$= -\frac{k}{2\pi} e^{-\frac{r^2}{4\nu t}}$$

Integrating,

$$rf = -\frac{k}{2\pi} e^{-\frac{r^2}{4\nu t}} + \text{constant}$$

When $r = 0$, f is finite and

$$\text{constant} = \frac{k}{2\pi}$$

$$\therefore rf = -\frac{k}{2\pi} e^{-\frac{r^2}{4\nu t}} + \frac{k}{2\pi}$$

$$= \frac{k}{2\pi}\left[1 - e^{-\frac{r^2}{4\nu t}}\right]$$

$$\Rightarrow f = \frac{k}{2\pi r}\left[1 - e^{-\frac{r^2}{4\nu t}}\right]$$

Thus

$$u = -\frac{y}{r} f(r,t) = -\frac{ky}{2\pi r^2}\left(1 - e^{-\frac{r^2}{4\nu t}}\right)$$

and $v = \frac{x}{r} f(r,t) = \frac{kx}{2\pi r^2}\left(1 - e^{-\frac{r^2}{4\nu t}}\right)$

#