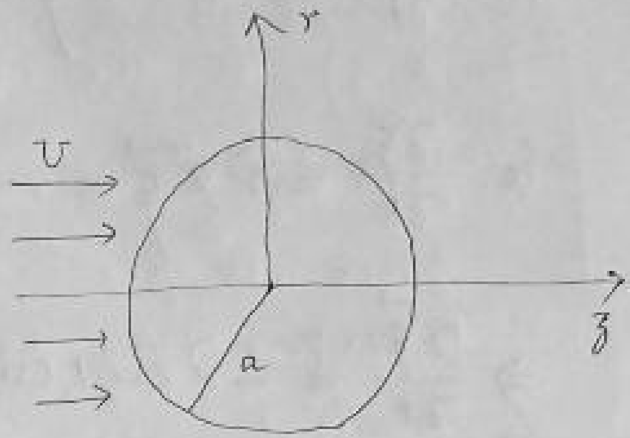
Study the cone of slow streaming past a sphere:



Suppose the liquid is moving past a sphere of radius 'a' very slowly, velocity at large distance from the centre of the sphere is U in the direction of z.

Let the cylindrical polar coordinates (r, θ, z) be taken at the centre of the sphere. We study the motion in the meridian plane rz i.e, θ = constant in the meridian plane.

Again R, ϕ be the plane polar coordinates in the meridian plane, then

$$r = R\sin\phi, \quad z = R\cos\phi .$$

Therefore the point in the meridian plane can be represented by (r, z) or by (R, ϕ) connected by the above relations. When the motion is steady, pressure is a function of (r, z) or (R, ϕ). We use stream function ψ in the meridian plane. $\left[\psi = \psi(r, z) \text{ or } \psi = \psi(R, \phi)\right]$

Here the governing equation of motion is

$$\frac{\partial \vec{q}}{\partial t} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{q}$$

Taking curl of it, $\text{curl}\left(\frac{\partial \vec{q}}{\partial t}\right) = \text{curl}\left(-\frac{1}{\rho}\nabla p\right) + \text{curl}\left(\nu \nabla^2 \vec{q}\right)$

i.e. $\dfrac{\partial \vec{\xi}}{\partial t} = 0 + \nu \nabla^2 \vec{\xi}$, where $\vec{\xi} = \text{curl}\,\vec{q}$

i.e. $\dfrac{\partial \vec{\xi}}{\partial t} = \nu \nabla^2 \vec{\xi}$

$$\Rightarrow \frac{\partial \vec{\xi}}{\partial t} = -\nu\, \text{curl curl}\, \vec{\xi} \longrightarrow \text{(iii)}$$

$$\left[\begin{aligned} \text{curl curl}\, \vec{\xi} &= \nabla \times (\nabla \times \vec{\xi}) \\ &= \nabla(\nabla \cdot \vec{\xi}) - \nabla^2 \vec{\xi} = -\nabla^2 \vec{\xi} \end{aligned} \right]$$

Now, when the motion is symmetrical about θ i.e. when velocity, pressure etc. are independent of θ, then there is one component of ~~velocity~~ vorticity. so, we find only one solution of vorticity. vorticity is perpendicular to the meridian plane. Therefore (iii) can be written as

$$\hat{i}_\theta \frac{\partial \xi_\theta}{\partial t} = -\nu \nabla \times \nabla \times (\hat{i}_\theta \xi_\theta) \qquad ; \quad \left| \, | \hat{i}_\varpi, \hat{i}_\theta, \hat{i}_z | \right.$$

Boundary condition for the motion is

(i) velocity is U at infinity in the position z-direction.

(ii) Velocity of the particles on the surface of the sphere is zero.

[Generally, ξ_θ is expressed in terms of stream function ψ in the meridian plane and the boundary conditions we applied in terms of the derivative of ψ.

If u, w be the velocity components in the direction of r, z respectively, then

$$u = \frac{1}{r}\frac{\partial \psi}{\partial z}, \quad w = -\frac{1}{r}\frac{\partial \psi}{\partial r};$$

$$w.r = -\frac{\partial \psi}{\partial r} \Rightarrow \psi = -w.\frac{r^2}{2}$$

In terms of polar coordinates in the meridian plane

$$q_R = \text{Velocity along } R = -\frac{1}{R^2 \sin\phi}\frac{\partial \psi}{\partial \phi}$$

$$q_\phi = \text{Velocity along } \phi = \frac{1}{R\sin\phi}\frac{\partial \psi}{\partial R}$$

[As the boundary conditions can be satisfied easily in terms of (R, ϕ), so ξ_θ is expressed in terms of ψ and R, ϕ coordinates.]

Now, $$\vec{\xi} = \frac{1}{R^2 \sin\phi}\begin{vmatrix} \hat{i}_R & R\,\hat{i}_\phi & R\sin\phi\,\hat{i}_\lambda \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \lambda} \\ q_R & R\,q_\phi & R\sin\phi\, q_\lambda \end{vmatrix}$$

$$= \frac{1}{R^2 \sin\phi}\begin{vmatrix} \hat{i}_R & R\,\hat{i}_\phi & R\sin\phi\,\hat{i}_\lambda \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \lambda} \\ -\frac{1}{R^2 \sin\psi}\frac{\partial \psi}{\partial \phi} & \frac{1}{\sin\phi}\frac{\partial \psi}{\partial R} & 0 \end{vmatrix}$$

(R, ϕ, λ) is the spherical polar coordinates
(r, θ, z) is the cylinderical polar coordinates
spherical λ = cylinderical θ
Since all quantities are independent of θ, therefore they are independent of λ also.

$$\vec{\xi} = \frac{1}{R^2 \sin\phi}\left[0.\hat{i}_R + 0.\hat{i}_\phi + R\sin\phi\left\{\frac{\partial}{\partial R}\left(\frac{1}{\sin\phi}\frac{\partial \psi}{\partial R}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{R\sin\phi}\frac{\partial \psi}{\partial \phi}\right)\right\}\hat{i}_\lambda\right]$$

$$= \hat{i}_\lambda \frac{1}{R}\left[\frac{1}{\sin\phi}\frac{\partial^2 \psi}{\partial R^2} + \frac{1}{R^2 \sin\psi}\frac{\partial^2 \psi}{\partial \phi^2} - \frac{1}{R^2}\frac{\cos\phi}{\sin^2\phi}\frac{\partial \psi}{\partial \phi}\right]$$

$$= \hat{i}_\lambda \frac{1}{R\sin\phi}\left[\frac{\partial^2 \psi}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2 \psi}{\partial \phi^2} - \frac{\cot\phi}{R^2}\frac{\partial \psi}{\partial \phi}\right]$$

$$= \hat{i}_\lambda \frac{1}{R\sin\phi}\left(D^2 \psi\right), \text{ where } D^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2}{\partial \phi^2} - \frac{\cot\phi}{R^2}\frac{\partial}{\partial \phi}$$

$$\therefore \text{curl}\, \vec{\xi} = \frac{1}{R^2 \sin\phi}\begin{vmatrix} \hat{i}_R & R\,\hat{i}_\phi & R\sin\phi\,\hat{i}_\lambda \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \lambda} \\ 0 & 0 & R\sin\phi\left(\frac{D^2\psi}{R\sin\phi}\right) \end{vmatrix}$$

$$= \frac{1}{R^2 \sin\phi}\left[\hat{i}_R \frac{\partial}{\partial \phi}\left(D^2 \psi\right) - R\,\hat{i}_\phi \frac{\partial}{\partial R}\left(D^2 \psi\right)\right]$$

$$\operatorname{curl}\operatorname{curl}\vec{\xi} = \frac{1}{R^2\sin\phi}\begin{vmatrix} \hat{i}_R & R\,\hat{i}_\phi & R\sin\phi\,\hat{i}_\lambda \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial\lambda} \\ \left(\frac{\frac{\partial}{\partial\phi}(D^2\psi)}{R^2\sin\phi}\right) & R\left(-\frac{\frac{\partial}{\partial R}(D^2\psi)}{R\sin\phi}\right) & R\sin\phi\cdot 0 \end{vmatrix}$$

$$= \frac{1}{R^2\sin\phi}\Big[0\cdot\hat{i}_R + R\,\hat{i}_\phi\cdot 0 + R\sin\phi\,\hat{i}_\lambda\Big\{\frac{\partial}{\partial R}\left(\frac{-1}{\sin\phi}\frac{\partial}{\partial R}D^2\psi\right)$$

$$-\frac{\partial}{\partial\phi}\left(\frac{1}{R^2\sin\phi}\frac{\partial}{\partial\phi}D^2\psi\right)\Big\}\Big]$$

$$= -\frac{\hat{i}_\lambda}{R}\left[\frac{\partial}{\partial R}\left(\frac{1}{\sin\phi}\frac{\partial}{\partial R}D^2\psi\right) - \frac{\partial}{\partial\phi}\left(\frac{1}{R^2\sin\phi}\frac{\partial}{\partial\phi}D^2\psi\right)\right]$$

$$= -\frac{\hat{i}_\lambda}{R}\Big[\frac{1}{\sin\phi}\frac{\partial^2}{\partial R^2}D^2\psi + \frac{1}{R^2\sin\phi}\frac{\partial^2}{\partial\phi^2}(D^2\psi)$$

$$-\frac{1}{R^2}\frac{\cos\phi}{\sin^2\phi}\frac{\partial}{\partial\phi}(D^2\psi)\Big]$$

$$= \frac{-\hat{i}_\lambda}{R\sin\phi}\left[\frac{\partial^2}{\partial R^2}(D^2\psi) + \frac{1}{R^2}\frac{\partial^2}{\partial\phi^2}(D^2\psi) - \frac{1}{R^2}\frac{\cos\phi}{\sin\phi}\frac{\partial}{\partial\phi}(D^2\psi)\right]$$

$$= -\frac{\hat{i}_\lambda}{R\sin\phi}\left[\frac{\partial^2}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2}{\partial\phi^2} - \frac{1}{R^2}\cot\phi\frac{\partial}{\partial\phi}\right](D^2\psi)$$

$$= -\frac{\hat{i}_\lambda}{R\sin\phi}D^2(D^2\psi) = -\frac{\hat{i}_\lambda}{R\sin\phi}(D^4\psi)$$

Therefore the vorticity equation

$$\frac{\partial\vec{\xi}}{\partial t} = -\nu\operatorname{curl}\operatorname{curl}\vec{\xi} \quad \text{becomes}$$

$$\frac{\partial}{\partial t}\left(D^2\psi\right) = \nu\left[D^4\psi\right]$$

In steady case, we have,

$$D^4\psi = 0$$

Boundary conditions are

(i) velocity is U at infinity

(ii) velocity is zero on the surface of the sphere.

To solve the equation $D^4\psi = 0$, we have put

$\mu = \cos\phi$ then,

$$\frac{\partial}{\partial\phi} = \frac{\partial}{\partial\mu}\frac{d\mu}{d\phi} = -\sin\phi\,\frac{\partial}{\partial\mu}$$

$$\frac{\partial^2}{\partial\phi^2} = \frac{\partial}{\partial\phi}\left(-\sin\phi\,\frac{\partial}{\partial\mu}\right)$$

$$= -\cos\phi\,\frac{\partial}{\partial\mu} - \sin\phi\left(-\sin\phi\,\frac{\partial^2}{\partial\mu^2}\right)$$

$$= -\cos\phi\,\frac{\partial}{\partial\mu} + \sin^2\phi\,\frac{\partial^2}{\partial\mu^2}$$

$$D^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2}{\partial\phi^2} - \frac{\cot\phi}{R^2}\frac{\partial^2}{\partial\phi^2}$$

$$= \frac{\partial^2}{\partial R^2} + \frac{1}{R^2}\left[-\mu\frac{\partial}{\partial\mu} + (1-\mu^2)\frac{\partial^2}{\partial\mu^2}\right] + \frac{1}{R^2}\sin\phi\cot\phi\,\frac{\partial}{\partial\mu}$$

$$= \frac{\partial^2}{\partial R^2} - \frac{\mu}{R^2}\frac{\partial}{\partial \mu} + \frac{1}{R^2}\frac{\partial^2}{\partial \mu^2} - \frac{\mu^2}{R^2}\frac{\partial^2}{\partial \mu^2} + \frac{\mu}{R^2}\frac{\partial}{\partial \mu}$$

$$\Rightarrow D^2 = \frac{\partial^2}{\partial R^2} + \frac{1-\mu^2}{R^2}\frac{\partial^2}{\partial \mu^2}$$

$$\therefore D^4 \psi = 0$$

$$\Rightarrow \left(\frac{\partial^2}{\partial R^2} + \frac{1-\mu^2}{R^2}\frac{\partial^2}{\partial \mu^2}\right)\left(\frac{\partial^2}{\partial R^2} + \frac{1-\mu^2}{R^2}\frac{\partial^2}{\partial \mu^2}\right)\psi = 0 \rightarrow (A)$$

We take $\psi = (1-\mu^2) f(R)$; $\psi = -U\frac{R^2}{2}$

so that

$$f(R) \sim -\frac{R^2}{2}U \text{ when } R \to \infty$$

Now, $D^2\psi = \left(\frac{\partial^2}{\partial R^2} + \frac{1-\mu^2}{R^2}\frac{\partial^2}{\partial \mu^2}\right)(1-\mu^2) f(R)$

$$= (1-\mu^2)\frac{\partial^2}{\partial R^2} f(R) + \frac{1-\mu^2}{R^2} f(R) \frac{\partial^2}{\partial \mu^2}(1-\mu^2)$$

$$= (1-\mu^2) f''(R) + \frac{1-\mu^2}{R^2} f(R)(-2)$$

$$= (1-\mu^2)\left[f''(R) - \frac{2f}{R^2}\right]$$

$$= (1-\mu^2)\left[\frac{d^2 f}{dR^2} - \frac{2f}{R^2}\right]$$

$$D^4\psi = \left[\frac{\partial^2}{\partial R^2} + \frac{1-\mu^2}{R^2}\frac{\partial^2}{\partial \mu^2}\right](1-\mu^2)\left[\frac{d^2 f}{dR^2} - \frac{2f}{R^2}\right]$$

$$= (1-\mu^2)\left[\frac{d^2}{dR^2} - \frac{2}{R^2}\right]\left[\frac{d^2 f}{dR^2} - \frac{2f}{R^2}\right]$$

$\therefore D^4\psi = 0$

$$\Rightarrow (1-\mu^2)\left[\frac{d^2}{dR^2} - \frac{2}{R^2}\right]\left[\frac{d^2}{dR^2} - \frac{2}{R^2}\right] f(R) = 0$$

$$\Rightarrow \left[\frac{d^2}{dR^2} - \frac{2}{R^2}\right]\left[\frac{d^2}{dR^2} - \frac{2}{R^2}\right] f(R) = 0, \because 1-\mu^2 \neq 0.$$

This equation is to be solved with the boundary conditions

$$f(R) \sim -\frac{R^2 U}{2}, \quad R \longrightarrow \infty$$

and $f(R) = f'(R) = 0$ at $R = a$.

The equation in $f(R)$ is an ordinary differential equation. Its solution is

$$f = \frac{A}{R} + BR + CR^2 + ER^4.$$

where A, B, C, E are four constants.

$$\therefore \psi = (1-\mu^2) f(R)$$

$$= \sin^2\phi \left[\frac{A}{R} + BR + CR^2 + ER^4\right] \longrightarrow \text{(vi)}$$

At infinity, $\psi = \sin^2\phi \, f(R)$

$$\psi = -\frac{R^2 U}{2}\sin^2\phi$$

$\therefore$ (vi) $\Rightarrow$ (at infinity)

$$-\frac{R^2 U}{2}\sin^2\phi = \sin^2\phi\left[\frac{A}{R} + BR + CR^2 + ER^4\right]$$

$$\Rightarrow -\frac{R^2 U}{2} = \frac{A}{R} + BR + CR^2 + ER^4$$

$$\Rightarrow -\frac{U}{2} = \frac{A}{R^3} + \frac{B}{R} + C + ER^2$$

$R \to \infty$, $-\frac{U}{2} = C + ER^2$

$$\therefore C = -\frac{U}{2} , E = 0$$

Again, at $R = a$, $\psi = \frac{\partial \psi}{\partial R} = 0$

$\therefore$ From (iv)

$$0 = \frac{A}{a} + Ba - \frac{1}{2} U . a^2$$

$$\left[\frac{\partial \psi}{\partial R} = \right] \quad 0 = -\frac{A}{a^2} + B - \frac{1}{2} \cdot 2 Ua$$

$$\therefore \quad A + Ba^2 = \frac{U}{2} a^3$$

$$A - Ba^2 = -\frac{U}{1} \cdot a^3$$

$$\therefore \quad 2A = \frac{Ua^3}{2} - Ua^3 = -\frac{1}{2} Ua^3 .$$

$$\Rightarrow A = -\frac{a^3 U}{4}$$

$$\therefore Ba^2 = -\frac{a^3 U}{4} + Ua^3 = \frac{3 \, Ua^3}{4} \Rightarrow B = \frac{3}{4} Ua .$$

$$\therefore \psi = \left[\frac{A}{2} + BR + CR^2 + ER^4\right] \sin^2 \phi$$

$$= -\frac{U}{2} \left[1 - \frac{3}{2} \frac{a}{R} + \frac{1}{2} \frac{a^3}{R^3}\right] R^2 \sin^2 \phi$$

$$q_R = -\frac{1}{R^2 \sin\phi} \frac{\partial \psi}{\partial \phi}$$

$$= -\frac{1}{R^2 \sin\phi}\left[-\frac{U}{2}\left\{1 - \frac{3}{2}\frac{a}{R} + \frac{1}{2}\frac{a^3}{R^3}\right\} R^2 . 2\sin\phi\cos\phi\right]$$

$$\Rightarrow q_R = U\cos\phi\left[1 - \frac{3}{2}\frac{a}{R} + \frac{1}{2}\frac{a^3}{R^3}\right]$$

$$q_\phi = \frac{1}{R\sin\phi}\frac{\partial \psi}{\partial R} = -\frac{1}{R\sin\phi}\frac{U}{2}\left[1 + \frac{3}{2}\frac{a}{R^2} - \frac{3}{2}\frac{a^3}{R^4}\right] R^2 \sin^2\phi$$

$$= U\sin\phi\left[\frac{3a}{2R} + \frac{1}{2}\frac{a^3}{R^3} - 1\right]$$

Hence,

Putting, $R \cos\phi = z$, $R\sin\phi = r$

$$q_r = \frac{3Urza}{4R^3}\left(\frac{a^2}{R^2} - 1\right)$$

$$q_z = U\left[\left(1 - \frac{3a}{4R} - \frac{1}{4}\frac{a^3}{R^3}\right) + \frac{3az^2}{4R^3}\left(\frac{a^2}{R^2} - 1\right)\right], \quad R^2 = r^2 + z^2.$$

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