$$= -e^{-\mu}\left[\frac{1}{2}f'' + \frac{1}{4}f'^2 + \frac{3}{2}\frac{f'}{r} - \frac{1}{2}\ddot{g}e^{\mu} - \frac{3}{4}\dot{g}^2 e^{\mu}\right]$$

$$R_3^3 = g^{33}R_{33} = -\frac{e^{-\mu}}{r^2\sin^2\theta} \cdot R_{22}\sin^2\theta$$

$$= -e^{-\mu}\left[\frac{1}{2}f'' + \frac{1}{4}f'^2 + \frac{3}{2}\frac{f'}{r} - \frac{1}{2}\ddot{g}e^{\mu} - \frac{3}{4}\dot{g}^2 e^{\mu}\right]$$

$$= R_2^2$$

$$R_4^4 = g^{44}R_{44} = \frac{3}{4}\ddot{g} + \frac{3}{4}\dot{g}^2$$

$$\therefore \quad R = R_j^i = R_1^1 + R_2^2 + R_3^3 + R_4^4$$
$$= -2e^{-\mu}\left(f'' + 2\frac{f'}{r} + \frac{1}{4}f'^2\right) + 3(\ddot{g} + \dot{g}^2)$$

Einstein's field equations are given by

$$R_j^i - \frac{1}{2}g_j^i R + \Lambda g_j^i = -8\pi T_j^i$$

$$\therefore \quad -8\pi T_1^1 = R_1^1 - \frac{1}{2}R + \Lambda$$

$$\Rightarrow 8\pi T_1^1 = -e^{-\mu}\left(\frac{f'}{r} + \frac{1}{4}f'^2\right) + \ddot{g} + \frac{3}{4}\dot{g}^2 - \Lambda \longrightarrow (7)$$

$$-8\pi T_2^2 = R_2^2 - \frac{1}{2}R + \Lambda$$

$$\Rightarrow 8\pi T_2^2 = -e^{-\mu}\left(\frac{1}{2}f'' + \frac{1}{2}\frac{f'}{r}\right) + \ddot{g} + \frac{3}{4}\dot{g}^2 - \Lambda \longrightarrow (8)$$

$$-8\pi T_3^3 = R_3^3 - \frac{1}{2}R + \Lambda \Rightarrow 8\pi T_3^3 = 8\pi T_2$$

$$\Rightarrow 8\pi T_3^3 = -e^{-\mu}\left(\frac{1}{2}f'' + \frac{1}{2}\frac{f'}{r}\right) + \ddot{g} + \frac{3}{4}\dot{g}^2 - \Lambda \longrightarrow (9)$$

$$-8\pi T_4^4 = R_4^4 - \frac{1}{2}R + \Lambda$$

$$\Rightarrow 8\pi T_4^4 = -e^{-\mu}\left(f'' + 2\frac{f'}{r} + \frac{1}{4}f'^2\right) + \frac{3}{4}\dot{g}^2 - \Lambda \longrightarrow (10)$$

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and $\quad 8\pi T^{i}_{j} = 0 \quad$ for $i \neq j$

The assumption of spatial isotropy of 3-space requires that

$$T_1^{\,1} = T_2^{\,2} = T_3^{\,3}$$

$\therefore$ from (7) and (8), we have

$$8\pi T_1^{\,1} = 8\pi T_2^{\,2}$$

$$\Rightarrow -e^{-\mu}\left(\frac{1}{2}\nu'' - \frac{1}{2}\frac{\nu'}{r} - \frac{1}{4}\nu'^2\right) = 0$$

$$\Rightarrow \nu'' - \frac{\nu'}{r} - \frac{1}{2}\nu'^2 = 0$$

$$\Rightarrow \frac{\nu''}{\nu'} = \frac{1}{r} + \frac{\nu'}{2}$$

Integration yields

$$\log \nu' = \log r + \frac{1}{2}\nu + \log k_1$$

$$\Rightarrow \nu' = k_1 r e^{\frac{1}{2}\nu}$$

$$\Rightarrow \frac{d\nu}{dr} = k_1 r e^{\frac{1}{2}\nu}$$, where k_1 is constant of integration

$$\Rightarrow e^{-\frac{1}{2}\nu} d\nu = k_1 r\, dr$$

Integrating we get

$$\frac{e^{-\frac{1}{2}\nu}}{-\frac{1}{2}} = k_1\frac{r^2}{2} - 2k_2$$ where k_2 is a constant of integration.

$$\Rightarrow e^{-\frac{1}{2}\nu} = -k_1\frac{r^4}{4} + k_2$$

$$= k_2\left(1 - \frac{k_1}{k_2}\frac{r^2}{4}\right)$$

$$\Rightarrow e^{\nu} = \frac{k_2^{1/2}}{\left(1 - \frac{k_1}{k_2}\frac{r^2}{4}\right)^2}$$

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Putting $-\frac{k_1}{k_2} = \frac{k}{R_0^2} \longrightarrow (11)$

where R_0^2 is a constant, which may be positive, negative or infinite, we can express the line-element as

$$ds^2 = dt^2 - \frac{e^{g(t)}}{\left(1+\frac{k}{4}\frac{r^2}{R_0^2}\right)^2}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right) \rightarrow (12)$$

where $k = +1, 0, -1$ corresponding to whether the constant R_0^2 given by (11) is positive, infinite or negative

This completes the derivation of Friedmann - Robertson - Walker (FRW) line-element.

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Dynamical consequences of FRW model:→

The Friedmann–Robertson–Walker (FRW) line-element is given by

$$ds^2 = dt^2 - e^{\mu(r,t)}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right) \longrightarrow (1)$$

with $\mu(r,t) = f(r) + g(t)$ and $e^{f(r)} = \dfrac{1}{\left(1+\frac{K}{4}\frac{r^2}{R_0^2}\right)^2}$

where $K = +1, 0, -1$ corresponding to whether the constant R_0^2 is positive, infinite or negative.

Assuming the universe to the field with a highly ideal fluid corresponding to an isotropic and homogeneous distribution of matter having ~~averg~~ average density ρ_0 and average presure P_0, in comoving coordinate system with $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, $x^4 = t$, the energy momentum tensor T^i_j given by $T^i_j = (P_0 + \rho_0)u^i u_j - g^i_j P_0 \longrightarrow (2)$ has the component

$$\left.\begin{aligned} &T^1_1 = -P_0, \quad T^2_2 = -P_0, \quad T^3_3 = -P_0 \\ &T^4_4 = \rho_0, \quad \text{and} \quad T^i_j = 0 \quad \text{for } i \neq j \end{aligned}\right\} \longrightarrow (3)$$

Therefore, for the line element (1) the Einstein field equations

$$R^i_j - \frac{1}{2} g^i_j R + \Lambda g^i_j = -8\pi T^i_j \longrightarrow (4)$$

gives

$$8\pi(-P_0) = -e^{-\mu}\left(\frac{f'}{r} + \frac{1}{4}f'^2\right) + \ddot{g} + \frac{3}{4}\dot{g}^2 - \Lambda \longrightarrow (5)$$

$$8\pi(-P_0) = -e^{-\mu}\left(\frac{1}{2}f'' + \frac{1}{2}\frac{f'}{r}\right) + \ddot{g} + \frac{3}{4}\dot{g}^2 - \Lambda \longrightarrow (6)$$

$$8\pi\rho_0 = -e^{-\mu}\left(f'' + 2\frac{f'}{r} + \frac{1}{4}f'^2\right) + \frac{3}{4}\dot{g}^2 - \Lambda \longrightarrow (7)$$

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