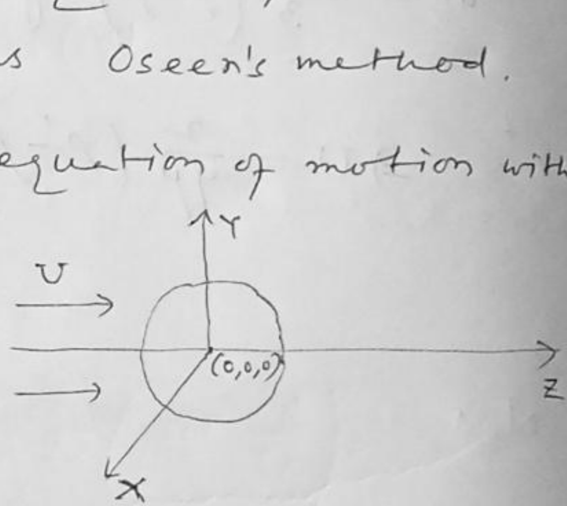
Oseen's approximation:

Oseen's approximation was given in the following way. The term $(\vec{q}\cdot\vec{\nabla})\vec{q}$ is negligible in comparison to the term $\nu\nabla^2\vec{q}$ (or $\frac{1}{R}\nabla^2\vec{q}$) [in the equation of motion] near the sphere when viscosity is mere prominent. But outside the sphere the term $(\vec{q}\cdot\nabla)\vec{q}$ and $\nu\nabla^2\vec{q}$ both are small quantities and they are of comparable magnitudes. So, one term cannot be neglected in comparison to the other. Stoke's approximation is not valid outside the sphere (or at a distance from the sphere). So, he gave a solution of the stoke's problem ~~xxx~~ partially the convective term in the equation of motion. His method of solution is known as Oseen's method.

Let us linearise the equation of motion with transformation

$$u = u'$$
$$v = v'$$
$$w = U + w'$$



Such that u', v', w' and the derivatives are small quantities, so, quares and products of which can be neglected. Here u, v, w are velocity component along x, y, z direction respectively, and U is the velocity along z-axis at infinity.

Now in steady case the equation of motion and continuity are

$$\left.\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u \\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v \\ u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} &= 0 \end{aligned}\right\} \quad (i)$$

substituting the expression for u, v, w in (1)

$$u'\frac{\partial u'}{\partial x}+v'\frac{\partial u'}{\partial y}+(U+w')\frac{\partial u'}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u'$$

$$u'\frac{\partial v'}{\partial x}+v'\frac{\partial v'}{\partial y}+(U+w')\frac{\partial v'}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v'$$

$$u'\frac{\partial (U+w')}{\partial x}+v'\frac{\partial}{\partial y}(U+w')+(w+U)\frac{\partial}{\partial z}(U+w') = -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2(U+w')$$

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial}{\partial z}(U+w') = 0$$

Neglecting the term of small orders, we have,

$$\left.\begin{aligned} U\frac{\partial u'}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u' \\ U\frac{\partial v'}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v' \\ U\frac{\partial w'}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w' \\ \frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z} &= 0 \end{aligned}\right\} \quad (ii)$$

In this case acceleration terms are partially obtained.
This are to be solved under the condition

$$u' = v' = w' = 0 \text{ at infinity } (R \to \infty)$$

and $u' = 0,\ v' = 0,\ w' = -U$ on the surface of the sphere $(R = a)$ $\Big\} \rightarrow$ (iii)

Now, if we differentiate the 1st equation of motion by 'x', 2nd equation w.r. to 'y' and the third equation w.r. to 'z' and add, and also use the equation of continuity then we get,

$$\nabla^2 p = 0 \quad \longrightarrow \text{(iv)}$$

$\therefore$ pressure satisfies the Laplace's equation.

Let us suppose

$$p = \rho U \frac{\partial \phi}{\partial z} \quad \text{where } \nabla^2 \phi = 0$$

and
$$\left.\begin{aligned} u' &= u'' - \frac{\partial \phi}{\partial x} \\ v' &= v'' - \frac{\partial \phi}{\partial y} \\ w' &= w'' - \frac{\partial \phi}{\partial z} \end{aligned}\right\} \longrightarrow \text{(v)}$$

using (v) in (ii), we have,

$$U\left[\frac{\partial}{\partial z}\left(u'' - \frac{\partial \phi}{\partial x}\right)\right] = -\frac{1}{\rho}\frac{\partial}{\partial x}\left(\rho U \frac{\partial \phi}{\partial z}\right) + \nu \nabla^2\left(u'' - \frac{\partial \phi}{\partial x}\right)$$

$$\Rightarrow U\left[\frac{\partial u''}{\partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right] = -U\frac{\partial^2 \phi}{\partial x \partial z} + \nu \nabla^2 u'' - \nu \nabla^2\left(\frac{\partial \phi}{\partial x}\right)$$

$$\Rightarrow U\frac{\partial u''}{\partial z} = \nu \nabla^2 u'' \quad ; \quad \nu \nabla^2\left(\frac{\partial \phi}{\partial x}\right) = \nu \frac{\partial}{\partial x}\left(\nabla^2 \phi\right) = 0$$

$$U\left[\frac{\partial}{\partial z}\left(v'' - \frac{\partial \phi}{\partial y}\right)\right] = -\frac{1}{\rho}\frac{\partial}{\partial y}\left(\rho U \frac{\partial \phi}{\partial z}\right) + \nu \nabla^2\left(v'' - \frac{\partial \phi}{\partial y}\right)$$

$$\left.\begin{array}{l} \text{i.e.} \quad U\dfrac{\partial v''}{\partial z} = \nu \nabla^2 v'' \\ \text{and} \quad U\dfrac{\partial w''}{\partial z} = \nu \nabla^2 w'' \\ \text{and} \quad \dfrac{\partial u''}{\partial x} + \dfrac{\partial v''}{\partial y} + \dfrac{\partial w''}{\partial z} = 0 \end{array}\right\} \rightarrow \text{(vii)}$$

i.e.

$$\nabla^2 u'' - \frac{U}{\nu}\frac{\partial u''}{\partial z} = 0$$

$$\nabla^2 v'' - \frac{U}{\nu}\frac{\partial v''}{\partial z} = 0$$

$$\nabla^2 w' - \frac{U}{\nu}\frac{\partial w''}{\partial z} = 0$$

$$\frac{\partial u''}{\partial x} + \frac{\partial v''}{\partial y} + \frac{\partial w''}{\partial z} = 0$$

If we put $\dfrac{U}{\nu} = 2k$, then we have

$$\left.\begin{array}{l} \left(\nabla^2 - 2k\dfrac{\partial}{\partial z}\right) u'' = 0 \\ \left(\nabla^2 - 2k\dfrac{\partial}{\partial z}\right) v'' = 0 \\ \left(\nabla^2 - 2k\dfrac{\partial}{\partial z}\right) w'' = 0 \\ \dfrac{\partial u''}{\partial x} + \dfrac{\partial v''}{\partial y} + \dfrac{\partial w''}{\partial z} = 0 \end{array}\right\} \rightarrow \text{(viii)}$$

From this equation of motion we may write the equation vorticity as

$$\left(\nabla^2 - 2k\frac{\partial}{\partial z}\right)\vec{\xi} = 0 \longrightarrow (ix)$$

Since the motion is symmetrical about z-axis, therefore vorticity component along z-axis is zero.
We may write

$$\vec{\xi} = (\xi, \eta, 0)$$

We take $\xi = -\frac{\partial \chi}{\partial y}$, $\eta = \frac{\partial \chi}{\partial x}$

where χ is a unknown function to be determined.

Thus we get,

$$\left(\nabla^2 - 2k\frac{\partial}{\partial z}\right)\left(-\frac{\partial \chi}{\partial y}\right) = 0$$

and $$\left[\nabla^2 - 2k\frac{\partial}{\partial z}\right)\left(\frac{\partial \chi}{\partial y}\right) = 0$$

i.e. $$-\frac{\partial}{\partial y}\left[\nabla^2 - 2k\frac{\partial}{\partial z}\right]\chi = 0$$

$$\frac{\partial}{\partial x}\left[\nabla^2 - 2k\frac{\partial}{\partial z}\right]\chi = 0$$

These two equation can be written as

$$\left(\nabla^2 - 2k\frac{\partial}{\partial z}\right)\chi = f(z) \longrightarrow (x)$$

Neglecting the arbitrary function $f(z)$, we put the equation in the form

$$\left(\nabla^2 - 2k\frac{\partial}{\partial z}\right)\chi = 0 \longrightarrow (xi)$$

Again, $2k\dfrac{\partial u''}{\partial z} = \nabla^2 u''$

$$= \frac{\partial \eta}{\partial z} - \frac{\partial \zeta}{\partial y}$$

$$= \frac{\partial^2 \chi}{\partial x \partial z}$$

$$2k\frac{\partial v''}{\partial z} = -\frac{\partial \zeta}{\partial x} - \frac{\partial \xi}{\partial z} = \frac{\partial^2 \chi}{\partial y \partial z}$$

$$2k\frac{\partial w''}{\partial z} = \frac{\partial \xi}{\partial y} - \frac{\partial \eta}{\partial x} = -\frac{\partial^2 \chi}{\partial y^2} - \frac{\partial^2 \chi}{\partial x^2}$$

$$= \frac{\partial^2 \chi}{\partial z^2} - \nabla^2 \chi$$

$$= \frac{\partial^2 \chi}{\partial z^2} - 2k\frac{\partial \chi}{\partial z} \quad ; \text{from } (xi)$$

i.e. $2k\dfrac{\partial u''}{\partial z} = \dfrac{\partial^2 \chi}{\partial x \partial z}$

$$2k\frac{\partial v''}{\partial z} = \frac{\partial^2 \chi}{\partial y \partial z}$$

$$2k\frac{\partial w''}{\partial z} = \frac{\partial^2 \chi}{\partial z^2} - 2k\frac{\partial \chi}{\partial z}$$

Integrating,

$$\left.\begin{aligned} 2k u'' &= \frac{\partial \chi}{\partial x} \quad \text{i.e. } u'' = \frac{1}{2k}\frac{\partial \chi}{\partial x} \\ 2k v'' &= \frac{\partial \chi}{\partial y} \quad \text{i.e. } v'' = \frac{1}{2k}\frac{\partial \chi}{\partial y} \\ 2k w'' &= \frac{\partial \chi}{\partial z} - 2k\chi \quad \text{i.e. } w'' = \frac{1}{2k}\left[\frac{\partial \chi}{\partial z} - 2k\chi\right] \end{aligned}\right\} \rightarrow (xii)$$

Now, from (xi), we have

$$\left[\nabla^2 - 2k\frac{\partial}{\partial z}\right]\chi = 0$$

This can be written as

$$(\nabla^2 - k^2)\left(e^{-kz}\chi\right) = 0 \longrightarrow \text{(xiii)}$$

The solution of this equation is found i.e. when $(e^{-kz}\chi)$ is a function of R $\left[R^2 = x^2+y^2+z^2\right]$. So that it vanishes at infinity. The solution is of the form

$$e^{-kz}\chi = c\, e^{-kR}/R \quad , \quad c \text{ is a constant}$$

$$\therefore \quad \chi = \frac{c e^{k(z-R)}}{R} \longrightarrow \text{(xiv)}$$

Again ϕ assumed to be

$$\phi = \frac{A_0}{R} + A_1\frac{\partial}{\partial z}\left(\frac{1}{R}\right) + A_2\frac{\partial^2}{\partial z^2}\left(\frac{1}{R}\right) + \cdots \quad \text{(xv)}$$

So that ϕ satisfies the Laplace's equation and ϕ vanishes at infinity.

The constant $c, A_0, A_1, A_2, \cdots$ are to be evaluated from the condition

$$u = v = w = 0 \quad \text{at } R = a$$

Many more constant have been found in Drag formula in power of Re (Reynold number) and $Re^2 \log Re$ is found. We find have only four constants and find drag formula with two terms only.

Now, using the value of χ, we have

$$u = u' = u'' - \frac{\partial \phi}{\partial x}$$

$$= \frac{1}{2k}\frac{\partial \chi}{\partial x} - \frac{\partial \phi}{\partial x}$$

$$v = v' = v'' - \frac{\partial \phi}{\partial y}$$

$$= \frac{1}{2k}\frac{\partial \chi}{\partial y} - \frac{\partial \phi}{\partial y}$$

$$w = U + w' = U + w'' - \frac{\partial \phi}{\partial z}$$

$$= U - \frac{\partial \phi}{\partial z} + \frac{1}{2k}\frac{\partial \chi}{\partial z} - \chi$$

But $\chi = \dfrac{c\, e^{-k(R-z)}}{R}$

$$= c\left[\frac{1}{R} - k + \frac{kz}{R} + \frac{k^2}{2R}(R-z)^2 + \cdots\right]$$

$$\therefore \frac{\partial \chi}{\partial z} = c\left[\frac{\partial}{\partial z}\left(\frac{1}{R}\right) + k\left\{\frac{1}{R} + z\frac{\partial}{\partial z}\left(\frac{1}{R}\right)\right\} + \frac{k^2}{2}\left\{\frac{2(R-z)}{R}\left(\frac{z}{R}-1\right) + (R-z)^2\frac{\partial}{\partial z}\left(\frac{1}{R}\right) + \cdots\right]$$

$$= \frac{c}{2R}\left[-\frac{k}{R} + 2k^2 - \frac{2k^2 z}{R} + \frac{\partial}{\partial z}\left(\frac{1}{R}\right) + \frac{k}{R}\left(1 - \frac{z^2}{R^2}\right) + k^2\left(\frac{2z}{R} - 1 - \frac{2}{2R} - \frac{z^3}{2R^3}\right)\right]$$