Case - III :- values of roots m_1 & m_2 are distinct (i.e. $m_1 \neq m_2$) and they Differ by an Integer (i.e $m_1 - m_2$ = Integer)

Trial Sol: – $\boxed{y = \sum_{n=0}^{\infty} a_n x^{m+n}}$; Complete Sol. :– $\boxed{y = c_1 (y)_{m_1} + c_2 \left(\frac{\partial y}{\partial m}\right)_{m_2} \text{ if } m_1 > m_2}$

Q. 7. Solve the Bessel's Differential Eq. of order 2 in series.

OR

Solve the Bessel's Diff. Eq of order 2 in series

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - 4) y = 0$$

OR

Using Frobenius method solve the Diff. Eq. $x^2 y'' + x y' + (x^2 - 4) y = 0$ about the point $x = 0$

57

Sol:- As we know that, Bessel's Differential Eq. of order n is written as :-

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0$$

Here order $n=2$:-

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-4)y = 0 \quad \text{--- (1)}$$

Here :- $P_0(x) = x^2$; $P_1(x) = x$; $P_2(x) = x^2-4$

Now at Point $x=0$:- $P_0(0) = (0)^2 = 0 \rightarrow$ Singular Point $\downarrow$

$$\boxed{\lim_{x\to a}(x-a)\frac{P_1(x)}{P_0(x)}}$$

$$\lim_{x\to 0}\frac{\cancel{(x)}(\cancel{x})}{\cancel{x^2}} = \boxed{1}$$

Finite
Analytic

$$\boxed{\lim_{x\to a}(x-a)^2\frac{P_2(x)}{P_0(x)}}$$

$$\lim_{x\to 0}\frac{\cancel{(x^2)}(x^2-4)}{\cancel{x^2}} = \boxed{-4}$$

Finite

$\rightarrow$ Regular singular Point

Sol:- $x=0$ is a Regular Singular Point. (58)

Now by Frobenius Method :-

Let Assumed Solution :- $y = \sum_{n=0}^{\infty} a_n x^{m+n}$

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n-2}$$

Now, put the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in eq. (1), we get:-

$$x^2 \left[\sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n-2} \right] + x \left[\sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1} \right]$$

$$+ (x^2-4) \left[\sum_{n=0}^{\infty} a_n x^{m+n} \right] = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}(m+n)(m+n-1)\,a_n x^{m+n} + \sum_{n=0}^{\infty}(m+n)\,a_n x^{m+n} + \sum_{n=0}^{\infty} a_n x^{m+n+2}$$

$$- \sum_{n=0}^{\infty} 4\,a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[\underline{(m+n)(m+n-1)} + \underline{(m+n)} - 4\right] a_n x^{m+n} + \sum_{n=0}^{\infty} a_n x^{m+n+2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[(m+n)(m+n-\cancel{1}+\cancel{1}) - 4\right] a_n x^{m+n} + \sum_{n=0}^{\infty} a_n x^{m+n+2} = 0$$

$$\Rightarrow \boxed{\sum_{n=0}\left[(m+n)^2 - 4\right] a_n x^{m+n} + \sum_{n=0}^{\infty} a_n x^{m+n+2} = 0 \quad \text{--- (2)}}$$

Now put the coefficient of least power of x i.e x^m to zero

$$\left[(m+0)^2 - 4\right] a_0 = 0$$

$$\Rightarrow \quad m^2 - 4 = 0 \quad \because \quad a_0 \neq 0$$

$\Rightarrow \quad m^2 = 4$

$\Rightarrow \quad m = \pm 2$

$\Rightarrow$ So, $m_1 = 2$ and $m_2 = -2$ $\quad$ [Let $m_1 > m_2$]

Here:- $\quad m_1 \neq m_2$ and $m_1 - m_2 = 2 - (-2) = 4$ = Integer

$\Downarrow$

$\boxed{\text{Case III}}$

Now from Eq. 2 :-

$$\sum_{n=0}^{\infty} \left[(m+n)^2 - 4\right] a_n x^{m+n} + \sum_{n=0}^{\infty} a_{n-2} x^{m+n} = 0$$

$\uparrow$ Indexing

$$\Rightarrow \sum_{n=0}^{\infty} \left[(m+n+2)(m+n-2) a_n + a_{n-2}\right] x^{m+n} = 0$$

Now put coeffi. of x^{m+n} to zero :-

$$(m+n+2)(m+n-2)\,a_n + a_{n-2} = 0$$

$$(m+n+2)(m+n-2)\,a_n = -a_{n-2}$$

$$\boxed{a_n = \frac{-1}{(m+n+2)(m+n-2)}\,a_{n-2}} \quad \forall\, n \geq 2 \quad -③$$

Which is the required Recurrance Relation.

Now put $n=2$ in eq. ③ :- $a_2 = \dfrac{-1}{(m+4)(m)}\,a_0 \Rightarrow \boxed{a_2 = \dfrac{-1}{m(m+4)}\,a_0}$

Now, put the coeff. of x^{m+1} to zero from Eq. ② :-

$$\left[(m+1)^2 - 4\right] a_1 = 0$$

$$\boxed{a_1 = 0}$$

(62)

Now put $n=3$ in eq. ③ :-

$$a_3 = \frac{-1}{(m+5)(m+1)} a_1 \Rightarrow \boxed{a_3 = 0}$$

Now put $n=4$ in eq. ③ :-

$$a_4 = \frac{-1}{(m+6)(m+2)} a_2$$

$$= \frac{-1}{(m+2)(m+6)} \left[\frac{-1}{m(m+4)} a_0 \right]$$

$$\boxed{a_4 = \frac{1}{m(m+2)(m+4)(m+6)} a_0}$$

Now put $n=5$ in eq. ③ :-

$$a_5 = \frac{-1}{(m+7)(m+3)} a_3 \Rightarrow \boxed{a_5 = 0}$$

Now put the values of a_1, a_2, a_3, a_4 and a_5 in Assumed sol:-

$$y = \sum_{n=0}^{\infty} a_n x^{m+n}$$

$$= a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + a_4 x^{m+4} + a_5 x^{m+5} + \cdots \infty$$

$$= x^m \left[a_0 + (0)x + \left[\frac{-1}{m(m+4)} a_0\right] x^2 + (0)x^3 + \left[\frac{+1}{m(m+2)(m+4)(m+6)} a_0\right] x^4 + (0)x^5 + \cdots \infty \right]$$

$$\boxed{y = a_0 x^m \left[1 - \frac{x^2}{m(m+4)} + \frac{x^4}{m(m+2)(m+4)(m+6)} + \cdots \infty \right]} \quad -(4)$$

Now, As we know that complete solution in case - III is:- (64)

$$y = c_1 (y)_{m_1} + c_2 \left(\frac{\partial y}{\partial m}\right)_{m_2} \quad ; \quad m_1 > m_2$$

$$\boxed{y = c_1 (y)_{m=2} + c_2 \left(\frac{\partial y}{\partial m}\right)_{m=-2}} \quad \text{— (5)}$$

Now put $m=2$ in eq. (4), we get:-

$$(y)_{m=2} = a_0 x^2 \left[1 - \frac{x^2}{(2)(6)} + \frac{x^4}{(2)(4)(6)(8)} + \text{———}\infty\right]$$

$$\boxed{(y)_{m=2} = a_0 x^2 \left[1 - \frac{x^2}{2 \cdot 6} + \frac{x^4}{2 \cdot 4 \cdot 6 \cdot 8} + \text{———}\infty\right]}$$

Put $a_0 = (m+2) b_0$

Note case III

$$y = b_0 x^m \left[(m+2) - \frac{(m+2) x^2}{m(m+4)} + \frac{x^4}{m(m+4)(m+6)} + \cdots \infty \right] \quad -⑥$$

Now Partially Diffr. eq. ⑥ w.r.t m, we get

$$\frac{\partial y}{\partial m} = b_0 (x^m \log x) \left[(m+2) - \frac{(m+2) x^2}{m(m+4)} + \frac{x^4}{m(m+4)(m+6)} + \cdots \infty \right]$$

$$+ b_0 x^m \left[1 - \frac{(m+2) x^2}{m(m+4)} \left[\frac{1}{m+2} - \frac{1}{m} - \frac{1}{m+4} \right] \right.$$

$$\left. + \frac{x^4}{m(m+4)(m+6)} \left[\frac{-1}{m} - \frac{1}{m+4} - \frac{1}{m+6} \right] + \cdots \infty \right]$$

$$\left(\frac{\partial y}{\partial m}\right)_{m=-2} = b_0 (x^{-2} \log x) \left[0 - 0 + \frac{x^4}{(-2)(2)(4)} + \cdots \infty \right]$$

$$+ b_0 x^{-2}\left[1 - 0 + \frac{x^4}{(-2)(2)(4)}\left[\cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} - \frac{1}{4}\right] + \text{———} \infty\right]$$

$$\boxed{\left(\frac{\partial y}{\partial m}\right)_{m=-2} = \frac{b_0 \log x}{x^2}\left[-\frac{x^4}{2\cdot 2\cdot 4} + \text{———}\infty\right] + \frac{b_0}{x^2}\left[1 + \frac{x^4}{2\cdot 2\cdot 4\cdot 4} + \text{———}\infty\right]}$$

Now put $(y)_{m=2}$ and $\left(\frac{\partial y}{\partial m}\right)_{-2}$ in complete solution (5) :- we get

$$y = c_1\left[a_0 x^2\left(1 - \frac{x^2}{2\cdot 6} + \frac{x^4}{2\cdot 4\cdot 6\cdot 8} + \text{———}\infty\right)\right]$$

$$+ c_2\left[\frac{b_0 \log x}{x^2}\left(-\frac{x^4}{2\cdot 2\cdot 4} + \text{———}\infty\right) + \frac{b_0}{x^2}\left(1 + \frac{x^4}{2\cdot 2\cdot 4\cdot 4} + \text{———}\infty\right)\right]$$

$$y = Ax^2\left(1 - \frac{x^2}{2\cdot 6} + \frac{x^4}{2\cdot 4\cdot 6\cdot 8} + \text{———} \infty\right)$$

$$+ B\left[\frac{\log x}{x^2}\left(-\frac{x^4}{2\cdot 2\cdot 4} + \text{———}\infty\right) + \frac{1}{x^2}\left(1 + \frac{x^4}{2\cdot 2\cdot 4\cdot 4} \text{———}\infty\right)\right]$$

$\textcircled{2}$

Here :- $A = c_1 a_0$, $B = c_2 b_0$

which is the required complete sol. of given D.E ①.