Statistical Mechanics
Lecture 11

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Bose-Einstein Condensation:

The derivation from perfect gas behaviour exhibited by Bose-Einstein gas is termed as gas degeneracy. The gas degeneracy is a function of $\frac{1}{D}$, where $D = e^{x}$, so we get

$$\frac{1}{D} = \frac{n}{g_s V} \left(\frac{2nmKT}{h^2}\right)^{-3/2} \rightarrow (i)$$

We note that at temperature nearing zero i.e. $T \approx 0$, $\frac{1}{D}$ will have a large value which means the degeneracy will be more marked. It means the gas will deviate highly from its perfect gas behaviour

The reason is that while at the Bose-Einstein distribution we have assumed that because of the closeness of the energy we could replace the discrete distribution by a continuous distribution giving that

$$n = \int_{0}^{\infty} dn(\epsilon) \rightarrow (ii)$$

i.e. we have changed summation into integration. As long as the change in occupation number n_i from state to state is small compared to the number of particles in the ststes. However the temperature in an ideal Bose gas is lowered to zero, the particles will begin to crowd in a few levels and the above condition will be violated.

This means that when working at low temperature one must be careful in replacing the summation into integration. The number of particles lying between energy range $\epsilon \; and \; \epsilon + d\epsilon$ is given by

$$dn(\epsilon) = g_s \frac{g(\epsilon)d\epsilon}{De^{\epsilon/KT} - 1} \rightarrow (iii)$$

Where

$$g(\epsilon) = \frac{4\pi mV}{h^3}\sqrt{2m\epsilon}$$

We note that for ground level state $\epsilon = \epsilon_0 = 0, g(\epsilon) = 0$, while actually it should have been unity because there is one state at $\epsilon = 0$.

Therefore above distribution fails to give number of states at $\epsilon = 0$ while this state called the ground state is very important at low temperature because a large number of particles occupy it at such temperature. We further note that if $\epsilon \neq 0$ the above distribution holds good as $g(\epsilon) \neq 0$. therefore the above distribution still be applied for all states except ground state which should be treated separately. For a single state

$$n_i = \frac{g_i}{e^{\alpha+\beta\epsilon_i} - 1} \rightarrow (iv)$$

For ground state $\epsilon_i = \epsilon_0 = 0\ and\ g_i = 1$, Therefore the number of particles at ground state

$$n_i \rightarrow n_0 = \frac{1}{e^{\alpha} - 1} \rightarrow (v)$$

Therefore the total number of particle states we can write

$$n = n_0 + \int dn \rightarrow (vi)$$

$$n = n_0 + \int g_s \frac{4\pi mV}{h^3} \frac{\sqrt{(2m\epsilon)}d\epsilon}{D-1} \rightarrow (vii)$$

Taking $x = \frac{\epsilon}{KT}$

$$n = n_0 + \frac{2g_s Z_t}{\sqrt{\pi}} \int_0^\infty \frac{x^{1/2}\, dx}{De^x - 1} \rightarrow (viii)$$

Here

$$Z_t = \left(\frac{2\pi mKT}{h^2}\right)^{3/2} . V \rightarrow (ix)$$

Further

$$n = n_0 + g_s Z_t F_{3/_2}(\alpha) \rightarrow (x)$$

Where

$$F_{3/_2}(\alpha) = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{x^{1/_2} dx}{De^x - 1} \rightarrow (xi)$$

$$F_{3/_2}(\alpha) = \frac{\sqrt{\pi}}{2D} \frac{2}{\sqrt{\pi}} (1 + \frac{1}{2^{3/_2} D} + \frac{1}{3^{3/_2} D^2} + \cdots .) \rightarrow (xii)$$

When

$$\alpha = 0, \qquad D = e^x = 1$$

So that we can writ

$$F_{3/2}(0) = \frac{\sqrt{\pi}}{2}\frac{2}{\sqrt{\pi}}\left(1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \cdots.\right) = 2.612 \rightarrow (xii)$$

Let $T = T_0,\ when\ \alpha = 0\ \ or\ D = 1\ \ we\ get$

$$n = g_s(z_t)_{T=T_0} F_{3/2}(0)$$

Therefore

$$g_s(z_t)_{T=T_0} = \frac{n}{F_{3/2}(0)} \rightarrow (xiii)$$

Putting the value of g_s we have

$$n = n_0 + n\,\frac{Z_t}{(Z_t)_{T=T_0}}\,\frac{F_{3/2}(\alpha)}{F_{3/2}(0)} \rightarrow (xiv)$$

$$n = n_0 + n\,\frac{\left(\frac{2\pi mKT}{h^3}\right)^{3/2}}{\left(\frac{2\pi mKT_0}{h^3}\right)^{3/2}}\,\frac{F_{3/2}(\alpha)}{F_{3/2}(0)}$$

$$n = n_0 + n\left(\frac{T}{T_0}\right)^{3/2}\frac{F_{3/2}(\alpha)}{F_{3/2}(0)}$$

$$n' = n - n_0 = n\left(\frac{T}{T_0}\right)^{3/2} \frac{F_{3/2}(\alpha)}{F_{3/2}(0)} \rightarrow (xv)$$

Now with $D > 1$, $F_{3/2}(\alpha)$ will have less value than when $D = 1$ $i.e.$

$$F_{3/2}(\alpha) \cong_{x \neq 0} F_{3/2}(0)$$

This means that n' given by equation (xv) acquires its maximum value when $\alpha = 0$. Thus for $\alpha = 0$ the maximum number of particles occupying states above ground state is given by

$$n' = n\left(\frac{T}{T_0}\right)^{3/2}, \text{When } T < T_0 \rightarrow (xvi)$$

And rest of the particles given by

$$n_0 = n - n' = n\left[1 - \left(\frac{T}{T_0}\right)^{3/2}\right] \rightarrow (xvii)$$

when $T < T_0$,must condense into ground state. Thus from equation $(xvii)$ we note that as the temperature is lowered beginning at $T = T_0$ the molecules fall rapidly into the ground state. There is a short of condensation into this state. This phenomenon is known as Bose-Einstein Condensation. The temperature T_0 at which the Bose-Einstein Condensation begins to occur depends upon the density of gas. If we consider liquid helium to be a gas one can obtain a value of about of 3.14 K for T_0 .

It is found experimentally that liquid helium does undergo a rather unusual transition at 2.19 K. Below this temperature the liquid helium displays the properties of a super fluid. It is generally agreed that this transition in liquid helium is associated with a Bose-Einstein Condensation.

For above temperature above T_0 , (α) must decrese in order to keep $n' \leq (n_0 + n) \leq n$.

Therefore $T > T_0$, $n_0 = 0$

And

$$n' = n = gZ_t F_{3/2}(\alpha)$$

From equation (xv), when $n' = n$, we find that

$$n = n\left(\frac{T}{T_0}\right)^{3/2} \frac{F_{3/2}(\alpha)}{F_{3/2}(0)}$$

$$F_{3/2}(\alpha) = F_{3/2}(0)\left(\frac{T_0}{T}\right)^{3/2}$$

$$F_{3/2}(\alpha) = 2.612\left(\frac{T_0}{T}\right)^{3/2} \rightarrow (xviii)$$

Which when substituted in the expression for n gives the total number of particle states equal to

$$n = g_s . Z_t . 2.612. \left(\frac{T_0}{T}\right)^{3/2} \rightarrow (xix)$$

Liquid Helium:

As an application of Bose-Einstein Statistics one may investigate the qualitative nature of the super fluid transistion of liquid helium at 2.2K. Ordinary helium consists almost entirely of neutral atoms of the isotope ${}_{2}^{4}He$.

As the total angular momentum of these atoms is zero, so explanation about helium must within the Bose-Einstein Statistics. Helium exhibits peculiar properties at low temperature. It is found that

i) Helium gas at atmospheric pressure condenses at 4.3 K into a liquid of very low density about $0.124 {}^{gm}/{}_{cm^3}$ (Its critical temperature is 5.2 K

ii) Frther cooling to about 0.82 K does not freez it and it is believed that it remains liquid all the way down to absolute zero. The solid state of helium does not form unless it is subjected to an external pressure of at least 23 atmosphere.

iii) For He^4 in liquid phase there is another phase transition called $\lambda - transistion$, which divides the liquid state into two phases $He - I\ and\ He - II$, while liquifying helium noted that at about 2.2K density appeared to pass through an abrupt maximum and decreasing slightly there after. It is also revealed that critical temperature is at 2.186 K and that it represents a transition to a new state of matter known as liquid $He - II$. In liquid $He - II$ it is found that

a) Heat conductivity is very large of the order of 3.10^6 times greater

b) Co-efficient of viscosity gradually diminishes as the temperature is lowered and appears to be approaching zero at absolute zero tempature.

c) Specific Heat measurement shows that specific heat curve discontinuous at 2.186K. The shape of the specific heat curve resembles the shape of λ and therefore this peculiar transition is $\lambda - transistion$ and the discontinuity temperature 2.186K is called $\lambda - point$. Liquid helium state has no latent heat concluded that $HeI \rightrightarrows HeII\ at\ \lambda_T$ is a second order transition.

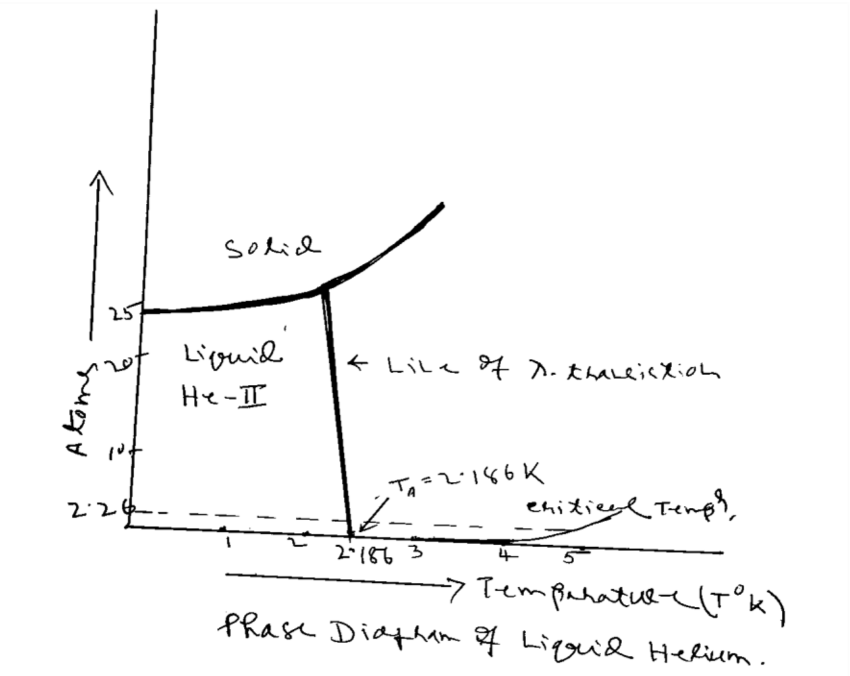

Fig 1

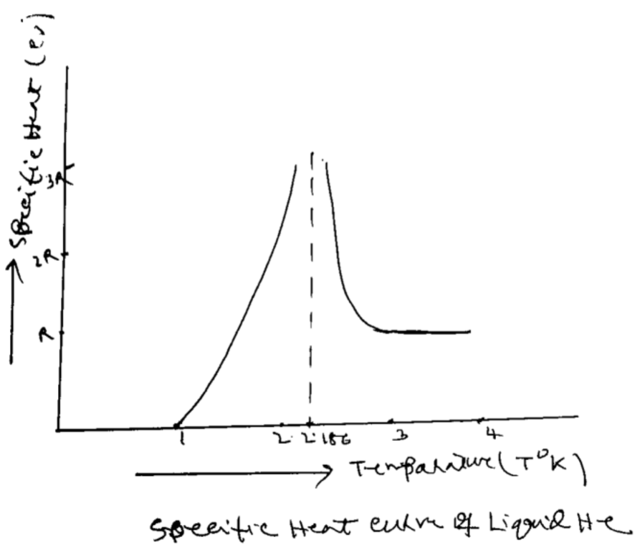

Fig 2