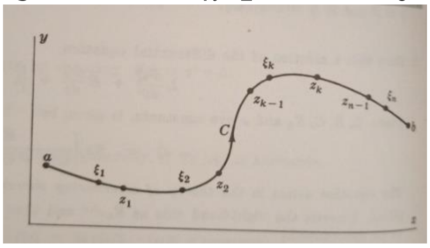
Complex Integration

Complex line integrals:

Let f(z) be continuous at all points of a curve C has a finite length. Let us subdivide C into n parts by means of points z_1 , $z_2, z_3 \ldots\ldots\ldots z_{n-1}$.$Let$ a= z_0 , b=z_n. Let us Choose a point ξ_k on each arc joining z_{k-1} to z_k (k=1,2,….n) .



Form the sum S_n = f(ξ_1) ($z_1 - a$)+ f(ξ_2) ($z_2 - z_1$) + f(ξ_1) ($z_3 - z_2$)+………. f(ξ_n) ($b - z_{n-1}$)

$$= \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1})$$

$$= \sum_{k=1}^{n} f(\xi_k)\,\Delta z_k \quad \text{where } \Delta z_k = z_k - z_{k-1}$$

When n become large Δz_k become small i.e as n → ∞ ,Δz_k → 0

Then the sum S_n approaches a fixed limit which does not depend on the mode of subdivision and we denote this limit by $\int_a^b f(z)dz$ or $\int_C \boldsymbol{f(z)dz}$

is called complex line integral along the curve C. Here f(z) is said to be integrable along C.

$$\int_a^b f(z)dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1})$$

Connection between real and complex line integral:

If f(Z) = u(x,y) + i v(x,y) = u+iv then

$$\int_C \boldsymbol{f(z)dz} = \int_C (\boldsymbol{u} + \boldsymbol{iv})(\boldsymbol{dx} + \boldsymbol{idy})$$

$$= \int_C \boldsymbol{udx} - \boldsymbol{vdy} + \int_C \boldsymbol{vdx} + \boldsymbol{udy}$$

Properties of Integral:

If f(z) and g(z) are integrable along C

1. $\int_C \{\boldsymbol{f(z)} + \boldsymbol{g(z)}\}\,\boldsymbol{dz} = \int_C \boldsymbol{f(z)dz} + \int_C \boldsymbol{g(z)dz}$

2. $\int_C \boldsymbol{k\,f(z)dz}$ **= k** $\int_C \boldsymbol{f(z)dz}$ **where k is a constant**

3. $\int_a^b f(z)dz = -\int_b^a f(z)dz$

4. $\int_a^b f(z)dz = \int_a^c f(z)dz + \int_c^b f(z)dz$ where a,b, c are on C

5. $\left|\int_a^b f(z)dz\right| \leq$ ML where $|f(z)| \leq$ M is an upper bound of $|f(z)|$ on C and L is the length of C

6. $\int_{c_1+c_2} \boldsymbol{f(z)dz} = \int_{c_1} \boldsymbol{f(z)dz} + \int_{c_2} \boldsymbol{g(z)dz}$ **where C=c_1+c_2**

Q. State and Prove Cauchy Integral theorem

Ans: Statement: Let f(z) = u(x,y) + iv(x,y) be analytic on and inside a simple closed contour and let $f^{/}(z)$ be also continuous on and inside C, then

$$\oint_c f(z)dz = 0$$

Proof: The proof of the Cauchy Integral theorem requires the Green Theorem for a positively oriented closed contour C; If the two real functions P(x,y) and Q(x,y) have continuous first order partial derivatives on and inside C, then

$$\oint_C Pdx + Qdy = \iint_D (Q_x - P_y)\,dxdy$$

Where D is the simply connected domain bounded by C.

Suppose we write f(z) = u(x,y) + iv(x,y) ; we have

$$\oint_c f(z)dz = \oint_c udx - vdy + i \oint_c vdx + u\,dy$$

One can infer from the continuity of $f^{/}(z)$ that u(x,y) and v(x,y) have continuous derivatives on and inside C. Using the Greens theorem the two real line integrals can be transformed into double integrals.

$$\oint_c f(z)dz = \iint_D (-v_x - u_y)dxdy + i \iint_D (u_x - v_y)dxdy$$

By Cauchy Rieman Equation

$u_x = v_y$ and $v_x = -u_y$

So $\iint_D (-v_x - u_y)dxdy = 0$ and $\iint_D (u_x - v_y)dxdy = 0$

Thus

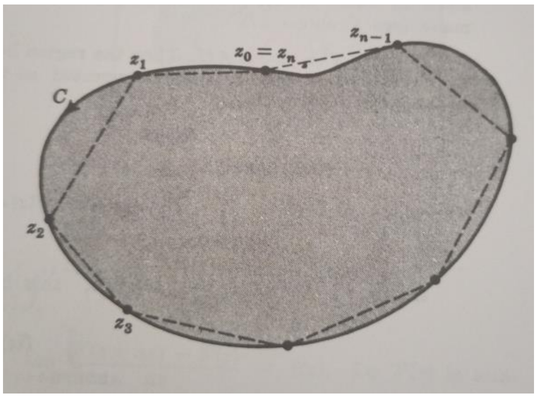
$$\oint_c f(z)dz = 0$$

Hence the theorem

Q.Prove Cauchy-Goursat theorem for any simple closed curve

Proof: Statement : If a function is analytic and one valued inside and on a simple closed curve C then

$$\oint_c f(z)dz = 0$$



Proof : Let us assume that C is contained in a region R in which f(z) is analytic.

Let $z_1, z_2, z_3, \ldots\ldots\ldots\ldots z_n$ be n subdivisions on curve where $z_0 = z_n$. Let us construct a polygon by joining these points.

Let us define the sum

$$S_n = \sum_{k=1}^{n} f(z_k)\Delta z_k$$

Where $\Delta z_k = z_k - z_{k-1}$. Since

$$\lim_{n\to\infty} S_n = \oint_c f(z)dz$$

It follows that given any $\varepsilon > 0$ we can choose N so that for n > N

$$\left|\oint_c f(z)dz - S_n\right| < \frac{\epsilon}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Now considering the integral along the polygon p. But by Cauchy -GGoursat theorem for any Closed polygon we have,

$\oint_p f(z)dz = 0$ Where p is the polygon

- ⇨ $\int_{z_0}^{z_1} f(z)dz + \int_{z_1}^{z_2} f(z)dz + \int_{z_2}^{z_3} f(z)dz + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \int_{z_{n-1}}^{z_n} f(z)dz = 0$
- ⇨ $\int_{z_0}^{z_1}\{f(z) - f(z_1) + f(z_1)\}\, dz + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \int_{z_{n-1}}^{z_n}\{f(z) - f(z_n) + f(z_n)\}dz = 0$
- ⇨ $\int_{z_0}^{z_1}\{f(z) - f(z_1)\}\, dz + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \int_{z_{n-1}}^{z_n}\{f(z) - f(z_n)\}dz + S_n = 0$
- ⇨ $S_n = \int_{z_0}^{z_1}\{f(z_1) - f(z)\}\, dz + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \int_{z_{n-1}}^{z_n}\{f(z_n) - f(z)\}dz \ldots\ldots\ldots\text{(ii)}$

Let us now choose N so large that on the line joining z_1 and z_1, z_1 and z_2,………. z_{n-1} and z_n,

$|f(z_1) - f(z)| < \frac{\epsilon}{2L} + |f(z_2) - f(z)| < \frac{\epsilon}{2L} + ...|f(z_n) - f(z)| < \frac{\epsilon}{2L}$ ....(iii)

Where L is the length of C. Then from (ii) and (iii) we have

$S_n \leq |\int_{z_0}^{z_1}\{f(z_1) - f(z)\}\, dz| + |\int_{z_0}^{z_1}\{f(z_1) - f(z)\}dz|$.........$+|\int_{z_{n-1}}^{z_n}\{ f(z_n) - f(z)\}dz|$

⇨ $|S_n| \leq \frac{\epsilon}{2L}\{|z_1 - z_0| + |z_2 - z_1| + |z_3 - z_2| + ...+|z_n - z_{n-1}|\} = \frac{\epsilon}{2}$.....(iv)

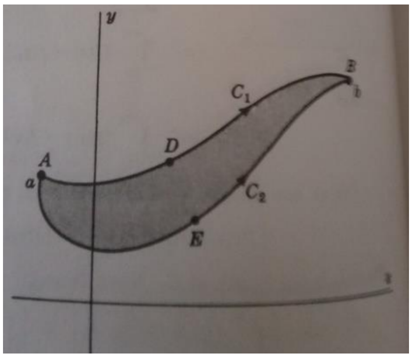
$$\oint_c f(z)dz = \oint_c f(z)dz - S_n + S_n$$

Now using (i) and (iv) , we have

$$|\oint_c f(z)dz| = |\oint_c f(z)dz - S_n| + |S_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since ϵ is arbitrary, it follows that $\oint_c f(z)dz = 0$

Q. If f(z) is analytic in a simply- connected region R, Prove that $\int_a^b f(z)dz$ is independent of the path in R joining any two points a and b in R



Solution: By Cauchy's theorem, We have

$\int_{ADBEA} f(z)dz = 0$

$\int_{ADB} f(z)dz + \int_{BEA} f(z)dz = 0$

⇨ $\int_{ADB} f(z)dz = -\int_{BEA} f(z)dz$

⇨ $\int_{ADB} f(z)dz = \int_{AEB} f(z)dz$

Thus $\int_{c_1} f(z)dz = \int_{c_2} f(z)dz = \int_a^b f(z)dz$

Q. If C is the curve y= $x^3 - 3x^2$ +4x -1 Joining Points (1,1) and (2,3) , find the value of

$$\int_C (12z^2 - 4iz)dz$$

Solution: Here $\int_C (12z^2 - 4iz)dz$ Here (1,1)=1+i and (2,3)=2+3i

$= \int_{1+i}^{2+3i}(12z^2 - 4iz)dz$

$= [4z^3 - 2iz^2]_{1+i}^{2+3i}$

$= 4[(2+3i)^3 - (1+i)^3] - 2i[(2+3i)^2 - (1+i)^2]$

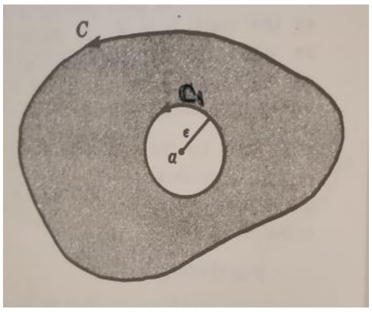
= 4[(8+36i- 54 –27i) – (1+3i-3 -i)] – 2i [(4 +12i -9) – (1+2i-1)]

= 4[(-44+ 7i) – 2i(-5 +10i)]

= -176 + 28i +10i +20

= - 156+38i

Q. Evaluate $\oint_c \frac{dz}{z-a}$ where C is any simple closed curve C and z=a is (a) outside C, (b) inside C



(a) If z=a outside the curve C then f(z) = $\frac{1}{z-a}$ is analytic everywhere inside and on C.

Hence by Cauchy's theorem

$$\oint_c \frac{dz}{z-a} = 0$$

(b) Suppose a is inside C. If z=a is inside C, then function is not analytic at z=a. Let us draw a circle c_1 surrounding the point z=a with radius ϵ with center at z=a so that c_1 is inside C .

$$\oint_c \frac{dz}{z-a} = \oint_{c_1} \frac{dz}{z-a}$$

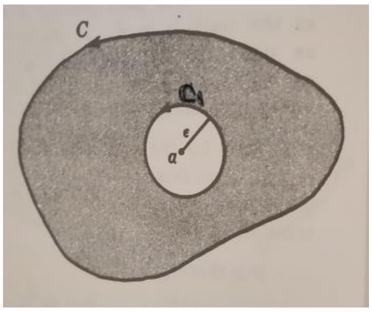
On c_1, $|z-a| = \epsilon$

⇨ $z-a = \epsilon\, e^{i\theta}$

⇨ $dz = \epsilon i\, e^{i\theta} d\theta$

$$\oint_{c_1} \frac{dz}{z-a} \qquad = \int_0^{2\pi} \frac{\epsilon i\, e^{i\theta} \mathrm{d}\theta}{\epsilon\, e^{i\theta}} \qquad = 2\,\pi i$$

Q. Evaluate $\oint_c \frac{dz}{(z-a)^n}$ n= 2, 3, 4,……… where z=a is inside the simple closed curve C



Solution: Suppose a is inside C. If z=a is inside C, then function is not analytic at z=a. Let us draw a circle c_1 surrounding the point z=a with radius ϵ with center at z=a so that c_1 is inside C .

$$\oint_c \frac{dz}{(z-a)^n} = \oint_{c_1} \frac{dz}{(z-a)^n}$$

On c_1, **|z-a| =** ϵ

z-a = $\epsilon\, e^{i\theta}$

⇨ dz = $\epsilon i\, e^{i\theta} \mathrm{d}\theta$

$$\oint_{c_1} \frac{dz}{(x-a)^n} = \int_0^{2\pi} \frac{\epsilon i\, e^{i\theta} \mathrm{d}\theta}{\epsilon^n\, e^{in\theta}} = \frac{i}{\epsilon^{n-1}} \int_0^{2\pi} e^{(1-n)i\theta} \mathrm{d}\theta$$

$$= \frac{i}{\epsilon^{n-1}} \left[\frac{e^{(1-n)i\theta}}{(1-n)i}\right]_0^{2\pi}$$

$$= \frac{i}{(1-\mathrm{n})\epsilon^{n-1}} \left[e^{2(1-n)\pi i} - 1\right]$$

$= \mathbf{0}$ **,** where n $\neq$ 1

[since $e^{2(n-1)\pi i} = \cos 2(n-1)\pi + \mathrm{i}\sin 2n\pi] = 1$ for all n

and $\frac{i}{(1-\mathrm{n})\epsilon^{n-1}}$ is not defined when n=1]

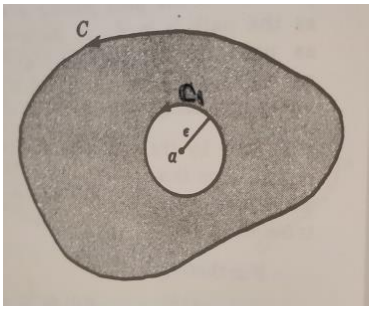
Q. **Evaluate (i) $\oint_c \frac{dz}{z}$ where C is a simple closed curve enclosing the origin.**

(ii) $\oint_c \frac{dz}{z}$ where C is a simple closed curve does not enclose the origin.

Q. **If f(z) is analytic inside and on boundary C of a simply-connected region R , prove that cauchy's integral formula**

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

Solution : The function $\frac{f(z)}{z-a}$ **is analytic inside and on C except at the point z=a .** Let us draw a circle c_1 surrounding the point z=a with radius ϵ with center at z=a so that c_1 is inside C .



$$\oint_c \frac{f(z)}{z-a}\,dz = \oint_{c_1} \frac{f(z)}{z-a}\,dz \;\ldots\ldots\ldots\text{(i)}$$

On c_1, |z-a| = ϵ

z-a = $\epsilon\, e^{i\theta}$

⇨ Z= a + $\epsilon\, e^{i\theta}$

⇨ dz = $\epsilon i\, e^{i\theta} d\theta$

$$\oint_{c_1} \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a+\epsilon\, e^{i\theta})\epsilon i\, e^{i\theta}}{\epsilon\, e^{i\theta}}\,d\theta$$

$$\oint_{c_1} \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a+\epsilon\, e^{i\theta})\,d\theta$$

Thus we have from (i)

$$\oint_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a+\epsilon\, e^{i\theta})\,d\theta$$

Taking the limit as $\epsilon \to 0$ **on both sides we have**

$$\oint_C \frac{f(z)}{z-a}\,dz = \lim_{\epsilon\to 0} i\int_0^{2\pi} f(a+\epsilon\, e^{i\theta})\,d\theta$$

$$\oint_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} \lim_{\epsilon\to 0} f(a+\epsilon\, e^{i\theta})\,d\theta$$

$$= \mathbf{i} \int_0^{2\pi} f(\mathrm{a})\, d\theta$$

$$\oint_C \frac{f(z)}{z-a}\, dz = 2\pi i\, \mathbf{f(a)}$$

Thus f(a) = $\frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz$

Q. If a function f(z) is analytic in a region D, then the derivative at any point z=a of D is also analytic in D , and given by

$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(x-a)^2} dz$ **,where C is any closed contour in D surrounding a point x=a**

Solution: Let a+h be a point in the neighourhood of the point a, then by Cauchy's integral formula , we have

$$\mathrm{f(a)} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz$$

So f(a+h) = $\frac{1}{2\pi i} \oint_C \frac{f(z+a)}{z-a-h}\, dz$

$$\mathbf{f(a+h)- f(a)} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a-h}\, dz - \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz$$

$$= \frac{1}{2\pi i} \oint_C \left(\frac{1}{z-a-h} - \frac{1}{z-a}\right) f(z) dz$$

$$= \frac{1}{2\pi i} \oint_C \left(\frac{(z-a)-(z-a-h)}{(z-a-h)(z-a)}\right) f(z) dz$$

$$= \frac{1}{2\pi i} \oint_C \left(\frac{hf(z)}{(z-a-h)(z-a)}\right. dz$$

$$\lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\right. = \lim_{h\to 0} \frac{1}{2\pi i} \oint_C \left(\frac{hf(z)}{(z-a-h)(z-a)}\right. dz$$

$$\Rightarrow f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz$$

Q. If a function f(z) is analytic in domain D, f(z) has, at any point z=a of D, derivatives of all orders, all of which are again analytic function in D, their values are given by

$$\boldsymbol{f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz}$$

Solution: Try yourself

Q. State and Prove Morera's theorem (Converse of Cauchy's theorem)

Solution: Statement: If f(z) is continuous in a simply connected region R and if

$$\oint_C f(z)dz = 0$$

around every simple closed curve C in R , then f(z) is analytic in R

Proof: $Let\ z_0$ be a fixed point and z any variable point in the Region R, then the value of the integral $\int_{z_0}^{z} f(z)dz$ is independent of the curve joining $z_0\ and$ z and is a function of z, hence we may write

$$F(z) = \int_{z_0}^{z} f(z)dz$$

$$\text{So } F(z+h) = \int_{z_0}^{z+h} f(z)dz$$

$$F(z+h) - F(z) = \int_{z_0}^{z+h} f(z)dz - \int_{z_0}^{z} f(z)dz$$

$$= \int_{z_0}^{z} f(z)dz + \int_{z}^{z+h} f(z)dz - \int_{z_0}^{z} f(z)dz$$

$$F(z+h) - F(z) = \int_{z}^{z+h} f(z)dz$$

$$\Rightarrow \lim_{h\to 0}[F(z+h) - F(z)] = \lim_{h\to 0} \int_{z}^{z+h} f(z)dz$$

$$= \oint_C f(z)dz = 0$$

where C is the Closed Curve, (when $h \to 0$ then

$$\int_{z}^{z+h} f(z)dz = \int_{z}^{z} f(z)dz = 0)$$

So, $\lim_{h\to 0}[F(z+h) - F(z)] = 0$

$$\lim_{h\to 0}[\frac{F(z+h)-F(z)}{h} - f(z)] = \lim_{h\to 0} \{-\frac{1}{h} f(z) \int_{z}^{z+h} dz\}$$

$$= \lim_{h\to 0} \{-\frac{1}{h} f(z) \int_C f(z)dz \}$$

Since Joining Curve C becomes closed when $h \to 0$

$$\lim_{h\to 0}[\frac{F(z+h)-F(z)}{h} - f(z)] = 0$$

$$=> \lim_{h\to 0}\frac{F(z+h)-F(z)}{h} = f(z)$$

$$=> F^{/}(z) = f(z)$$

Thus, derivative of F(z) exists for all z, therefore F(z) is analytic in R, consequently $F^{/}$(z) is analytic i.e f(z) is analytic.

Q. If f(z) is analytic within a circle C, given by |z-a| = R and if |f(z)|≤M on C, then $|f^n(a)| \leq \frac{Mn!}{R^n}$ (It is the Cauchy's Inequality.)

Solution: We know that if f(z) is analytic in a certain domain then derivative of all orders at any point z=a is also analytic and the nth derivative of f(z) at z=a is given by

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

$$|f^n(a)| = \left|\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz\right| \leq \frac{n!}{|2\pi i|}\int_C \frac{|f(z)||dz|}{|(z-a)|^{n+1}} \text{..........(i)}$$

Given |z-a| = R$e^{i\theta}$

Or, z-a = $Re^{i\theta}$
Or, dz = i$Re^{i\theta}d\theta$
Or, | dz| = | i$R^{i\theta}d\theta$|= R$d\theta$ (|i|=1,| $Re^{i\theta}| = R$

From (i) we have ,

$$|f^n(a)| \leq \frac{n!}{2\pi}\int_C \frac{MRd\theta}{R^{n+1}} \quad (\text{since } |f(z)| \leq M, \text{ given})$$

$$|f^n(a)| \leq \frac{n!}{2\pi}\frac{M}{R^{n+1}}\int_0^{2\pi} Rd\theta$$

$$\leq \frac{n!}{2\pi}\frac{M}{R^{n+1}}2\pi R$$

$$\leq \frac{Mn!}{R^n}$$

Q. Prove that derivative of an analytic function is analytic

Solution: Let f(z) be an analytic function in the domain D.
If C is any closed curve in D and z=a point within C.
Then

$$\boldsymbol{f^{/}(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz}\textbf{....................(i)}$$

If f(a + h) be a point in the neighourhood of the point a within C, then

Then $f^{/}(a+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a-h)^2}dz$

So, $f^{/}(a+h) - f^{/}(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a-h)^2}dz - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz$

$= \frac{1}{2\pi i}\int_C \{\frac{1}{(z-a-h)^2} - \frac{1}{(z-a)^2}\}f(z)dz$

$= \frac{1}{2\pi i}\int_C \{\frac{(z-a)^2-(z-a-h)^2}{(z-a-h)^2(z-a)^2}\}f(z)dz$

$= \frac{1}{2\pi i}\int_C \{\frac{2h(z-a-\frac{h}{2})}{(z-a-h)^2(z-a)^2}\}f(z)dz$

⇨ $\frac{f^{/}(a+h)-f^{/}(a)}{h} = \frac{2!}{2\pi i}\int_C \{\frac{(z-a-\frac{h}{2})}{(z-a-h)^2(z-a)^2}\}f(z)dz$

⇨ $\lim_{h\to 0}\frac{f^{/}(a+h)-f^{/}(a)}{h} = \frac{2!}{2\pi i}\int_C \{\frac{(z-a)}{(z-a)^2(z-a)^2}\}f(z)dz$

⇨ $f^{//}(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^3}$

⇨ $f^{/}(z)$ **is differentiable at any point z=a in the domain D**

⇨ $\boldsymbol{Hence}\ f^{/}(z)$ **is analytic.**

Q. If f(z) = $\frac{z^2+5z+6}{z-2}$ **. Does Cauchy theorem apply?**

(i) When the path of integration C is a circle of radius 3 with origin as centre.

(ii) When C is a circle of radius 1 with origin as centre.

Solution: Given f(z) = $\frac{z^2+5z+6}{z-2}$

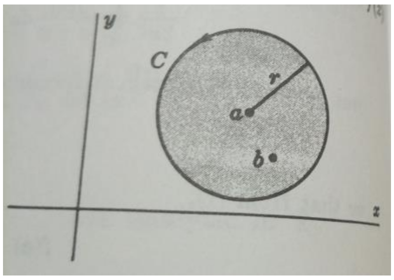
Clearly f(z) is not analytic at z=2

(i) When C is a circle with radius 3 then |z|=3 , the point |z|=2 lies inside it . Therefore , f(z) is not analytic within the circle, so in this case Cauchy's theorem does not apply i.e. $\int_C \frac{z^2+5z+6}{z-2}dz \neq \mathbf{0}$

(ii) When C is the circle with radius 1 then |z|=1, the point |z|=2 lies outside it , so that f(z) is analytic within and on the circle. Hence in this case Cauchy's theorem can be apply apply i.e. $\int_C \frac{z^2+5z+6}{z-2}dz = \mathbf{0}$

Q. State and Proof Liouville's Theorem

Solution : Statement: If for all z in the entire complex plane, (i) f(z) is analytic and (ii) f(z) is bounded i.e if there is a constant M such that $|f(z)| < M$, then f(z) must be a constant.



Proof: Let a and b be any two points in the z plane . Suppose that C is a circle of radius r having centre at a and enclosing point b.

From Cauchy's integral formula , we have

$$f(b) - f(a) = \frac{1}{2\pi i} \oint_C (\frac{f(z)}{(z-b)} dz\ dz - \frac{1}{2\pi i} \oint_C (\frac{f(z)}{(z-b)} dz$$

$$= \frac{b-a}{2\pi i} \oint_C (\frac{f(z)}{(z-b)(z-a)} dz$$ Type equation here.

Now we have

$$|z-a| = r,\ |z-b| = |(z-a) + (a-b)| \geq |z-a| - |a-b| = r - |a-b| \geq \frac{r}{2}$$

If we Choose r so large that $|a-b| < \frac{r}{2}$. Then since $|f(z)| < M$ and length of C is $2\pi r$, then we have

$$|f(b) - f(a)| = \frac{|b-a|}{2\pi i} \left| \oint_C (\frac{f(z)}{(z-b)(z-a)} dz \right| \leq \frac{|b-a| M (2\pi r)}{2\pi \left(\frac{r}{2}\right) r} = \frac{2|b-a|M}{r}$$

Let $r \to \infty$ we see that $|f(b) - f(a)| = 0$, which shows that f(z) must be a constant.

Q. Prove that every polynomial equation

$$P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots\ldots\ldots\ldots + a_n z^n = 0.$$

where the degree $n \geq 1$ and $a_n \neq 0$ has at least one root.