~~Neighbourhood of a point and Neighbourhood.~~

Neighbourhood of a point and Neighbourhood of a set:

Let (X, T) be a topological space and A be a subset of X. A is said to be a neighbourhood of a point $x \in X$ if there exist an open set G in T such that $x \in G \subset A$. Obviously any open set G containing x is also a neighbourhood of x.

A set $N \subset X$ is called a neighbourhood of a set $A \subset X$, if there exist a set G in T such that $A \subset G \subset N$. Every open set containing A is a neighbourhood of A.

Ex: Let $X = \{a, b, c, d\}$

$T = \{\phi, \{b\}, \{a, b, d\}, X, \{a, b\}\}$.

Then (X, T) is a topological space. Let us consider the point a. The subsets of A containing a are:
$\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$ and X.

$\because \{a, b\}$ is an open set containing a,

$\therefore \{a, b\}$ is a Neighbourhood of a.

Also $a \in \{a, b\} \subset \{a, b, c\}$ and $a \in \{a, b\} \subset \{a, b, d\}$

$\therefore$ the sets $\{a, b, c\}$ and $\{a, b, d\}$ are also neighbourhood of a.

The set X is a neighbourhood of every point in X.
$\therefore$ the neighbourhood of the point a are;
$\{a,b\}$, $\{a,b,c\}$, $\{a,b,d\}$ and X.

Note: Let (X,T) be a topological space and $x \in X$
Then,

1. there is at least one neighbourhood of x.

2. If N is a neighbourhood of x and $M \subset X$ such that $M \supset N$ then M is also a neighbourhood of x.

3. If N_1 and N_2 are two neighbourhood of x; then $N_1 \cap N_2$ is also a neighbourhood of x.

Interior point and Interior of a set:

Let (X,T) be a topological space and $A \subset X$. A point $x \in A$ is said to be an interior point of A if $\exists$ an ~~point~~ open set G in T such that $x \in G \subset A$. The set of all interior point of A, denoted by $Int(A)$ is called the interior of the set A. obviously the interior of A is the union of all open sets contained in A.

Note:
1. $Int(A) = A$ if and only if A is open.
2. $Int(A) \subset A \subset \bar{A}$
3. $Int(A)$ is the longest open set contained in A.
4. $Int(X-A)$ is called the exterior of A. It is denoted by $Ext(A)$. The points of exterior of A are called the exterior points of the set A.

Boundary of a set:

Let A be a subset of a topological space (X, T). The boundary of A, denoted by $b(A)$, is defined as the set $\bar{A} \cap \overline{(X-A)}$. Every point of boundary of A is called a boundary point of A.

Note:
1. $\bar{A} = Int(A) \cup b(A)$
2. $b(A)$ is a closed set.

No where dense set:

A subset A of a topological space (X, T) is said to be no where dense in X if the interior of the closure of A is empty. i.e $Int(\bar{A}) = \phi$.

Accumulation point or limit point or cluster point or derived point :

Let, (X, τ) be a topological space and A be a subset of X. A point $x \in X$ is said to be a limit point or an accumulation point of A if every open set G containing x contains a point of A different from x.

Ex: In a indiscrete topological space $(X, \mathcal{J})$, X and ϕ are the only open set in X. Hence X is the only open containing any $x \in X$. Let $A \subseteq X$.

1. If $A = \phi$, then A has no limit point.
2. If $A = \{x\}$, then every point of $X - \{x\}$ is a limit point of A.
3. If A contains more than one element of X, then every point of X is a limit point of A.

Derived set : Let A be a subset of a topological space (X, τ). The set of limit point of A, denoted by $D(A)$, is called the derived set of A.

Ex: In a indiscrete topological space $(X, \mathcal{J})$

$$D(A) = \begin{cases} \phi, & \text{if } A = \phi \\ X - \{x\}, & \text{if } A = \{x\},\ x \in X \\ X, & \text{if } A \text{ contains more than one point.} \end{cases}$$

Ex: Let $X = \{a, b, c, d, e\}$ and

$T = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$

Let $A = \{a, b, c\}$. Then find the $D(A)$, $Int(A)$, $Ext(A)$ and $b(A)$.

Soln: Let us first consider the point $a \in X$. Since $\{a\}$ is an open set containing 'a' which does not contain a point of A different from a, therefore a is not a limit point of A. Let us now consider the point $b \in X$. The open sets containing b are $\{b, c, d, e\}$ and X. Since every open set containing b contain a point of A different from b, so, b is a limit point of A.

Similarly 'c' is not a limit point of A. but d and e are limit point of A.

$\therefore D(A) = \{b, d, e\}$

Again, since $\{a\}$ is the only open set contained in A.

$\therefore Int(A)$ = the largest open set contained in A.

$\Rightarrow Int(A) = \{d, e\}$

Now $X - A = \{d, e\}$. Since $X - A$ has no interior point,

$\therefore \text{Int}(X - A) = \phi$

$\Rightarrow \text{Ext}(A) = \phi$

The closed sets in X are

X, $\{b, c, d, e\}$, $\{a, b, e\}$, $\{b, e\}$, $\{a\}$ and ϕ

$\therefore \bar{A} =$ intersection of all closed superset of A.

$\Rightarrow \bar{A} = X$

and $X - A = \{d, e\}$

$\therefore \overline{X - A} = X \cap \{b, c, d, e\} = \{b, c, d, e\}$.

$\therefore b(A) = \bar{A} \cap \overline{(X - A)}$

$= X \cap \{b, c, d, e\}$.

$= \{b, c, d, e\}$

<u>Ex</u>: Consider the topology

$T = \{\phi, \mathbb{N}, A_n = \{1, 2, 3, \ldots, n\} : n \in \mathbb{N}\}$ on the set $\mathbb{N}$ of natural numbers.

Find, 1. $D(A)$ if $A = \{3n : n \in \mathbb{N}\}$

2. $D(A)$ if $A = \{1, 4, 5, 7\}$

*Let (X, T) be a topological space and $A \subset X$. A point $x \in A$ is said to be an isolated point of A if it has a neighbourhood which contain no other point of A.

H.W

Theorem: Let β be a class of subsets of a non-empty set X. Then β is a base for some topology on X, if and only if it possesses the following two properties;

1) $X = \cup \{B : B \in \beta\}$

2) For any $B_1, B_2 \in \beta$, $B_1 \cap B_2$ is the union of member of β, or, equivalently, if $x \in B_1 \cap B_2$ then $\exists$ $B \in \beta$ such that ~~[illegible]~~ $x \in B \subseteq B_1 \cap B_2$.

Base for a topology:

Let (X, T) be a topological space. A class β of open subsets of X; i.e $\beta \subset T$ is a base for the topology T if and only if every open set $G \in T$ is the union of members of β.

Equivalently $\beta \subset T$ is a base for T if and only if for any point x belongs to an open set G, there exist $B \in \beta$ with $x \in B \subset G$.

Ex: The open intervals form a base for the usual topology on the line $\mathbb{R}$. For if $G \subset \mathbb{R}$ is open and and $x \in G$, then by definition $\exists$ an open intervals (a,b) with $x \in (a,b) \subset G$.

Similarly, the open discs form a base for the usual topology on the plane $\mathbb{R}^2$.

Ex: Consider any discrete space $(X, \mathcal{D})$. Then the class $\beta = \{\{x\} : x \in X\}$ of all singleton subsets of X is a base for the discrete topology on X.

Note: 1. Let β be a basis ~~and~~ for the topology T on a non empty set X. Then a subset G of X is open in X iff $\forall\, x \in G$, $\exists\, B \in \beta$ such that $x \in B \subset G$.

2. Let X be a non empty set and β be a basis for a topology T on X. Then T equals the collection of all unions of elements of β.

3. Let β be a basis for a topology T on a non empty set X. Since X is an open set, therefore $X = \bigcup\{B : B \in \beta\}$. That is union of all basis elements equals X.

Subbases:

Let (X, T) be a topological space. A class $\mathcal{S}$ of open subsets of X i.e $\mathcal{S} \subset T$, is a subbase for the topology T on X if and only if finite intersections of members of $\mathcal{S}$ form a base for T.

Theorem: Let β be a basis for a topology T on a non empty set X and Y be a subset of X. Show that the collection $\beta_Y = \{B \cap Y : B \in \beta\}$ is a basis for the relative topology on Y.

Proof: Given, β be a basis for a topology T on X and $Y \subset X$.

$\therefore$ The relative topology T_Y on Y is defined as

$T_Y = \{Y \cap G : G \in T\}$

Now, we have to show that the collection

$\beta_Y = \{B \cap Y : B \in \beta\}$ is a basis for the

relative topology T_Y on Y.

Let $y \in Y \cap G$, [$Y \cap G$ is an open set in Y]

$\Rightarrow y \in Y$ and $y \in G$

$\because G \in T$

$\therefore \exists$ a $B \in \beta$ such that $y \in B \subset G$

$\therefore y \in B \subset G$

$\Rightarrow y \in Y \cap B \subset Y \cap G$, [$\because y \in Y$]

$\Rightarrow y \in B \cap Y \subset Y \cap G$ and $B \cap Y \in \beta_Y$.

Hence, for each open set $Y \cap G$, if $y \in Y \cap G$, then there exist a set $B \cap Y$ in β_Y such that $y \in B \cap Y \subset Y \cap G$

$\therefore \beta_Y$ is a Basis for the relative topology T_Y on Y.

Continuous functions:

Let (X, T_1) and (Y, T_2) be two topological spaces. A function $f: X \to Y$ is said to be continuous if for ~~any~~ every open subset G of Y, the inverse image $f^{-1}(G)$ is an open subset of X.

Ex: Let $X = \{a, b, c, d\}$

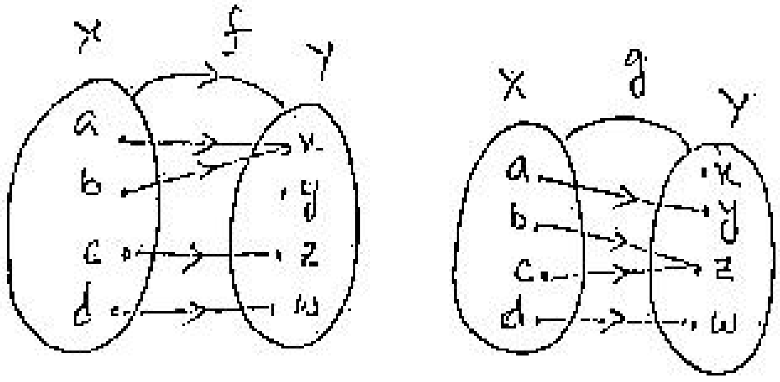
$$T_1 = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, X\}$$

and $Y = \{x, y, z, w\}$

$$T_2 = \{\phi, \{x\}, \{y\}, \{x, y\}, \{y, z, w\}, Y\}$$

Then (X, T_1) and (Y, T_2) are topological spaces.

Let $f: X \to Y$ and $g: X \to Y$ be defined by the diagrams:



Now the open subset of X are,

ϕ, $\{a\}$, $\{a, b\}$, $\{a, b, c\}$, X

and the open subset of Y are

ϕ, $\{x\}$, $\{y\}$, $\{x, y\}$, $\{y, z, w\}$, and Y

Also, $f^{-1}(\phi) = \phi$

$f^{-1}(\{x\}) = \{a, b\}$

$f^{-1}(\{y\}) = \phi$

$f^{-1}(\{x, y\}) = \{a, b\}$

$f^{-1}(\{y, z, \omega\}) = \{c, d\}$

$f^{-1}(Y) = X$

$\because f^{-1}(\{y, z, \omega\}) = \{c, d\}$ is not open in X.

$\therefore f : X \to Y$ is not continuous.

And $g^{-1}(\phi) = \phi$

$g^{-1}(\{x\}) = \phi$

$g^{-1}(\{y\}) = \{a\}$

$g^{-1}(\{x, y\}) = \{a\}$

$g^{-1}(\{y, z, \omega\}) = X$

$g^{-1}(Y) = X$

$\because g^{-1}(G)$ is open in X whenever G is open in Y.

$\therefore g : X \to Y$ is continuous.

Ex: Let $(X, \mathcal{D})$ be a discrete topological space and (Y, T) be any topological space. Then every function $f: X \longrightarrow Y$ is continuous.

Ex: Let (X, T) be any topological space and $(Y, \mathcal{I})$ be an indiscrete topological space. Then every function $f: X \longrightarrow Y$ is continuous.

Ex: Let (X, T_1) and (Y, T_2) be any to topological space then every constant function $f: X \longrightarrow Y$ is continuous.

Theorem: Let (X, T_1) and (Y, T_2) be two topological spaces. Then a function $f: X \longrightarrow Y$ is continuous if and only if for every closed set F of Y, the inverse image $f^{-1}(F)$ is closed in X.

Proof: Let $f: X \rightarrow Y$ be continuous. Let F be an arbitrary closed subset of Y. Then $Y - F$ is open in Y.

$\because$ $f: X \rightarrow Y$ is continuous

$\therefore$ $f^{-1}(Y - F)$ is open in X

But $f^{-1}(Y - F) = X - f^{-1}(F)$

$\therefore$ $X - f^{-1}(F)$ is open in X. This implies $f^{-1}(F)$

is closed in X. This shows that $f^{-1}(F)$ is closed in X whenever F is closed in Y.

Conversely let $f^{-1}(F)$ be closed in X whenever F is closed in Y. To show that $f: X \to Y$ is continuous, let G_1 be an arbitrary open subset of Y. Then $Y - G_1$ is closed in Y.

$\therefore$ $f^{-1}(Y - G_1)$ is closed in X.

$\Rightarrow$ $X - f^{-1}(G_1)$ is closed in X.

$\Rightarrow$ $f^{-1}(G_1)$ is open in X

Consequently, $f^{-1}(G_1)$ is open in X whenever G_1 is open in Y

$\therefore f: X \to Y$ is continuous.

<u>Theorem:</u> Let (X, T_1) and (Y, T_2) be two topological spaces and let β be a basis for the topology on Y. Then a function $f: X \to Y$ is continuous if and only if the inverse image of every member of the basis β is an open subset of X.

<u>Proof:</u> Let $f: X \to Y$ be continuous. Then every member of the basis β for the topology on Y being open in Y, the inverse image of every member of β is open in X.

Conversely let the inverse image of every member of the basis β be an open set in X. To show that $f: X \to Y$ is continuous, let G be an arbitrary open set in Y. Since β is a basis for the topology on Y, therefore G is of the form,

$$G = \bigcup_{\alpha} B_\alpha \quad \text{where } B_\alpha \in \beta$$

Now, $f^{-1}(G) = f^{-1}(\bigcup_{\alpha} B_\alpha) = \bigcup_{\alpha} f^{-1}(B_\alpha)$

$\because B_\alpha \in \beta, \ \forall \alpha$

$\therefore$ By our assumption, $f^{-1}(B_\alpha)$ is open in X, $\forall \alpha$

$\therefore \bigcup_{\alpha} f^{-1}(B_\alpha)$ is open in X.

So, consequently $f^{-1}(G)$ is open in X, whenever G is open in Y.

Hence $f: X \to Y$ is continuous.

<u>Theorem:</u> Let (X, T_1) and (Y, T_2) be two topological spaces and let $\mathcal{S}$ be a subbasis for the topology on Y. Then a function $f: X \to Y$ is continuous if and only if the inverse image of every member of subbasis $\mathcal{S}$ is an open set in X.

proof: Let $f: X \to Y$ be continuous. Let S be an arbitrary member of the subbasis $\mathcal{S}$ for the topology on Y.

$\because$ every member of the subbasis is an open set in Y, therefore $f^{-1}(S)$ is open in X, whenever $S \in \mathcal{S}$.

Conversely let the inverse image of every member of the subbasis $\mathcal{S}$ be an an open set in X. To show that $f: X \to Y$ is continuous, let G be an arbitrary open set in Y.

$\because$ $\mathcal{S}$ is a subbasis for the topology on Y.

$\therefore$ G is of the form,

$$G = \bigcup_i \left(S_{i1} \cap S_{i2} \cap S_{i3} \cap \cdots \cap S_{in} \right), S_{ik} \in \mathcal{S}.$$

$$\therefore f^{-1}(G) = f^{-1}\left\{ \bigcup_i \left(S_{i1} \cap S_{i2} \cap S_{i3} \cap \cdots \cap S_{in} \right) \right\}$$

$$= \bigcup_i f^{-1}\left(S_{i1} \cap S_{i2} \cap S_{i3} \cap \cdots \cap S_{in} \right)$$

$$= \bigcup_i \left\{ f^{-1}(S_{i1}) \cap f^{-1}(S_{i2}) \cap f^{-1}(S_{i3}) \cap \cdots \cap f^{-1}(S_{in}) \right.$$

But $S_{ik} \in \mathcal{S} \Rightarrow f^{-1}(S_{ik})$ is open in X

Consequently, $f^{-1}(G)$ is the arbitrary union of finite intersection of open subset of X.

$\because$ the arbitrary union and finite intersection of open sets is open.

$\therefore$ $f^{-1}(G)$ is an open subset of X.

This show that $f^{-1}(G)$ is open in X whenever G is open in Y.

Hence $f: X \to Y$ is continuous.

<u>Theorem</u>: Let (X, T_1) and (Y, T_2) be two topological spaces. Then a function $f: X \to Y$ is continuous if and only if for every subset A of X, $f(\bar{A}) \subset \overline{f(A)}$.

<u>proof</u>: Let $f: X \to Y$ be continuous and A be an arbitrary subset of X. Now,

$$f(A) \subset Y \text{ and } f(A) \subset \overline{f(A)}.$$

$\because$ $\overline{f(A)}$ is a closed set in Y and $f: X \to Y$ is continuous,

$\therefore$ $f^{-1}(\overline{f(A)})$ is closed in X containing A.

But, $A \subset f^{-1}(f(A))$

$\therefore$ $A \subset f^{-1}(\overline{f(A)})$ ... $[\because f(A) \subset \overline{f(A)}]$

$\Rightarrow$ $\bar{A} \subset f^{-1}(\overline{f(A)})$ $[\because \bar{A}$ is the smallest closed superset of $A]$

$\Rightarrow f(\bar{A}) \subset f\left(f^{-1}\left(\overline{f(A)}\right)\right)$

$\Rightarrow f(\bar{A}) \subset \overline{f(A)}$

Conversely let $f(\bar{A}) \subset \overline{f(A)}$, $\forall A \subset X$.

To show that $f^{-1}(F)$ is closed in X whenever F is closed in Y.

So, let F be a closed subset of Y and $A = f^{-1}(F)$.

Then A is a ~~closed~~ subset of X

$\therefore f(\bar{A}) \subset \overline{f(A)}$

But $\bar{A} = \overline{f^{-1}(F)} \Rightarrow f(\bar{A}) = f\left(\overline{f^{-1}(F)}\right)$

$\Rightarrow f(\bar{A}) \subseteq \overline{f(f^{-1}(F))}$

$\Rightarrow f(\bar{A}) \subseteq \bar{F}$

$\Rightarrow f(\bar{A}) \subseteq F$ [$\because F$ is closed]

$\therefore \bar{A} \subset f^{-1}(F)$

$\Rightarrow \bar{A} \subset A$

Also $A \subset \bar{A}$, $\forall A \subset X$

$\therefore$ we have,

$A = \bar{A}$

This means that A is a closed subset of X.
Consequently $f^{-1}(F)$ is closed in X whenever F is closed in Y.

Hence $f: X \to Y$ is continuous.

Continuity at a point:

Let (X, T_1) and (Y, T_2) be two topological spaces. A function $f: X \to Y$ is said to be continuous at a point $x_0 \in X$, if for every open subset V of Y containing $f(x_0)$, there exist an open subset U of X such that $x_0 \in U$ and $f(U) \subset V$.