16. PHASE VELOCITY

The phase velocity of a wave is the velocity with which a point of definite phase of the wave, such as its crest or trough is propagating through a medium.

A one-dimensional harmonic wave is described by a wave function $\psi(x, t)$ as

$$\psi(x, t) = A \sin(kx - \omega t + \phi) \quad ...(16.1)$$

where A is the amplitude, ω is the angular frequency, k is the wave number or propagation constant and ϕ is an initial phase, which is constant. The argument for the sine function $\theta(x,t) = kx - \omega t + \phi$ is called the phase of the wave. A surface of constant phase is known as a wave front.

Therefore, for a wave front, the phase

$$\theta(x,t) = kx - \omega t + \phi \text{ is constant} \quad ...(16.2)$$

If we assume that the initial phase $\phi = 0$ then $kx - \omega t = C$, where C is a constant.

Therefore, the equation of the wave front for a one-dimensional harmonic wave is

$$x = \frac{C + \omega t}{k}$$

This equation represents a plane parallel to the yz plane. Thus, the wave fronts for a one-dimensional harmonic wave are planes perpendicular to the direction of motion of wave and hence such a wave is also called plane wave.

Suppose that at the instant of time t, the wave profile is given by the dotted line and after an infinitesimal time δt later, the profile has been changed to solid line (Fig.16.1). The head (B) and tail (A) of an arrow represent the points of same phase. The propagation of a wave means the propagation of its phase.

Let the position and time co-ordinates for the point A be (x, t) and the co-ordinates for the point B be $(x + \delta x, t + \delta t)$. Since $(kx - \omega t)$ expresses the phase of a wave, therefore equality of phase at these two positions can be expressed as

$$[k(x + \delta x) - \omega(t + \delta t)] = (kx - \omega t)$$

$$\Rightarrow \quad k\,\delta x = \omega\,\delta t$$

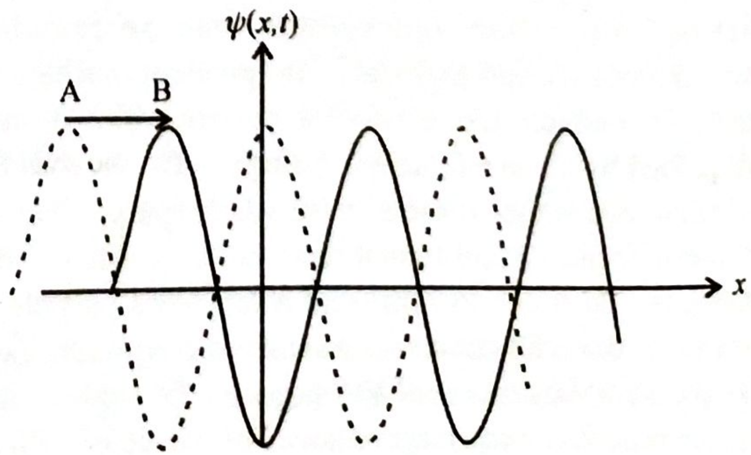

Fig. 16.1. Illustration of profile of the waveform.

$$\lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{\omega}{k}$$

The left hand side denotes the rate of change of position with time, *i.e.* the velocity with which the phase transports itself.

Thus phase velocity or wave velocity $v_p = \dfrac{\omega}{k}$ ...(16.3)

In terms of frequency and wavelength, it can be expressed as

$$v_p = \frac{\omega}{k} = \upsilon\lambda \qquad \left[\because \omega = 2\pi\upsilon \text{ and } k = \frac{2\pi}{\lambda}\right]$$

The phase velocity is also given in terms of wavelength λ and time period T of the wave.

$$v_p = \frac{\lambda}{T} \qquad \left[\because \upsilon = \frac{1}{T}\right] \quad ...(16.4)$$

17. GROUP VELOCITY

When two or more waves with slightly different velocity and frequency superimpose on each other, a group of waves or a wave packet is formed. The velocity with which the entire group of these waves travels is known as the group velocity. It is also referred as the velocity of the wave packet.

Consider the superposition of two waves having same amplitude A but slightly different frequencies ω_1 and ω_2 and wave numbers k_1 and k_2. The corresponding wave funcitons ψ_1 and ψ_2 are :

$$\psi_1 = A\cos(k_1x - \omega_1t) \quad ...(17.1)$$

$$\psi_2 = A\cos(k_2x - \omega_2t) \quad ...(17.2)$$

The respective phase velocities are $v_{p_1} = \dfrac{\omega_1}{k_1}$ and $v_{p_2} = \dfrac{\omega_2}{k_2}$

When the two waves superimpose, the resulting wave function will be

$$\psi = \psi_1 + \psi_2$$

$$\psi = A\cos(k_1x - \omega_1t) + A\cos(k_2x - \omega_2t)$$

Using the mathematical identity : $\cos A + \cos B = 2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$, we get

$$\psi = 2A\cos\left(\frac{(k_1+k_2)x - (\omega_1+\omega_2)t}{2}\right)\cos\left(\frac{(k_1-k_2)x - (\omega_1-\omega_2)t}{2}\right)$$

If we consider

$(\omega_1 + \omega_2)/2 = \omega$, the average value of angular velocity ;

$(k_1 + k_2)/2 = k$, the average value of propagation constant ;

$\omega_1 - \omega_2 = \Delta\omega$, the difference in value of angular velocity ;

$k_1 - k_2 = \Delta k$, the difference in value of propagation constant.

Then

$$\psi = 2A\cos\left(\frac{(\Delta kx - \Delta\omega t)}{2}\right)\cos(kx - \omega t) \quad ...(17.3)$$

The above wave function represents a resultant wave travelling with frequency ω, whose amplitude is given by

$$A(x, t) = 2A\cos\left(\frac{\Delta kx - \Delta\omega t}{2}\right) \quad ...(17.4)$$

The amplitude is no more a constant but depends on both position and time. Thus, the superposition of two waves results in an amplitude modulation of the total wave. The amplitude given by eqn. (17.4) can be interpreted as a broad oscillating envelope (Fig. 17.1) or wave group or wave packet, which has an angular frequency $\Delta\omega/2$ and propagation constant $\Delta k/2$. This envelope moves with a velocity

$$v_g = \frac{\Delta\omega}{\Delta k}$$

If ω and k have continuous spreads, then the group velocity becomes

$$v_g = \frac{d\omega}{dk} \quad ...(17.5)$$

When we superimpose waves of different amplitudes and frequencies, we obtain a wave packet which travels with the group velocity v_g as a whole.

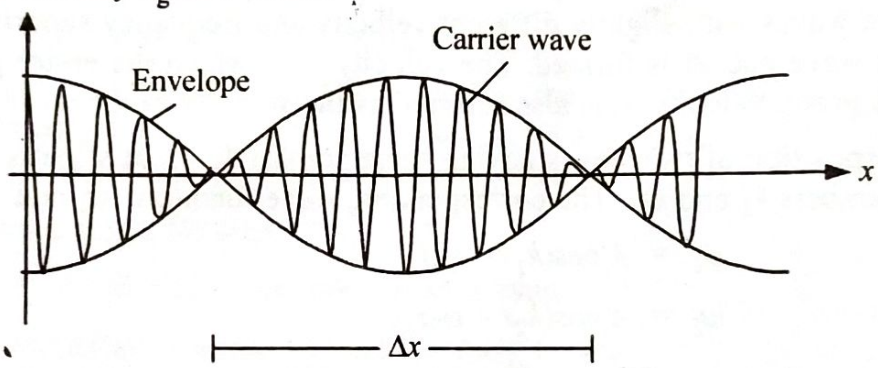

Fig. 17.1. The envelope which forms the wave packet travels with group velocity v_g and the carrier wave travels with phase velocity v_p.

Although the crests inside the wave packet move with the phase velocity, the envelope of the packet, moves with the group velocity given by

$$v_g = \frac{d\omega}{dk}$$

Considering the total wave function as given by eqn. (17.3), we observe that $\cos(kx - \omega t)$ represents the carrier wave which moves with a phase velocity $v_p = \frac{\omega}{k}$. This carrier wave lies inside an envelope, which modulates the amplitude of carrier wave.

18. RELATION BETWEEN THE PHASE VELOCITY AND GROUP VELOCITY

The phase velocity $\quad v_p = \frac{\omega}{k}$

$\therefore \quad \omega = kv_p$

Taking the derivative w.r.t. k on both sides of the above equation, we get

$$\frac{d\omega}{dk} = v_p + k\frac{dv_p}{dk}$$

$$\therefore \quad v_g = v_p + k\frac{dv_p}{dk} \quad ...(18.1)$$

In order to express this equation in terms of wavelength λ, let us use the relation

$$k = \frac{2\pi}{\lambda} \quad ...(18.2)$$

$$\therefore \quad dk = -\frac{2\pi}{\lambda^2}d\lambda \quad ...(18.3)$$

Dividing eqn. (18.2) by eqn. (18.3), we get

$$\frac{k}{dk} = -\frac{\lambda}{d\lambda} \quad ...(18.4)$$

Substituting eqn. (18.4) in eqn. (18.1), we get

$$v_g = v_p - \lambda\frac{dv_p}{d\lambda} \quad ...(18.5)$$

The above equation gives the desired relation between the group velocity and the phase velocity.

Case 1 : Non-Dispersive Medium : It is a medium in which waves of different wavelengths have same velocity, *i.e.* v_p is independent of λ, so

$$\frac{dv_p}{d\lambda} = 0$$

Hence, from the eqn. (18.5), we have

$$v_g = v_p$$

The group velocity is equal to the phase velocity for a non-dispersive medium. The shape of the group does not change as it propagates forward.

Case 2 : Dispersive Medium : It is a medium in which waves of different wavelengths move with different velocities, *i.e.*

$$\frac{dv_p}{d\lambda} \neq 0$$

$$\therefore \quad v_g \neq v_p$$

Key Points :

1. Group velocity is the velocity with which a wave packet travels.
2. The group velocity is always equal to the velocity of the particle. Hence, we can use a wave group to represent a material particle.

19. RELATION BETWEEN GROUP VELOCITY AND THE PARTICLE VELOCITY

The velocity of the wave packet v_g should be equal to the velocity of the material particle v. Let us prove their equality.

The angular velocity $\quad \omega = 2\pi\upsilon = \frac{2\pi E}{h}$

$$= \frac{2\pi mc^2}{h}$$

$$= \frac{2\pi m_0 c^2}{h\sqrt{1-\frac{v^2}{c^2}}}$$

where m_0 is the rest mass, h is the Planck constant and E is the total energy of the particle.

$$\therefore \quad \omega = \frac{2\pi m_0 c^2}{h}\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

Differentiating the above equation w.r.t. v on both sides, we get

$$\frac{d\omega}{dv} = \frac{2\pi m_0 c^2}{h}\left(-\frac{1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left(-\frac{2v}{c^2}\right)$$

$$= \frac{2\pi m_0 v}{h}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}$$

The propagation constant k can be expressed as

$$k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h} = \frac{2\pi mv}{h} = \frac{2\pi m_0 v}{h\sqrt{1-\frac{v^2}{c^2}}}$$

where p is the momentum of the particle and $\lambda = \frac{h}{p}$

$$k = \frac{2\pi m_0 v}{h}\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

Differentiating the above equation w.r.t. v on both sides, we get

$$\frac{dk}{dv} = \frac{2\pi m_0}{h}\left[\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} + v\left(-\frac{1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left(-\frac{2v}{c^2}\right)\right]$$

$$\frac{dk}{dv} = \frac{2\pi m_0}{h}\left[\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} + \left(\frac{v^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\right]$$

$$= \frac{2\pi m_0}{h}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left[\left(1-\frac{v^2}{c^2}\right)+\left(\frac{v^2}{c^2}\right)\right]$$

$$= \frac{2\pi m_0}{h}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}$$

Since the group velocity $v_g = \frac{d\omega}{dk}$

Dividing eqn. (19.1) by eqn. (19.2), we get

$$v_g = v$$

Therefore, the group velocity of the wave group associated with the material particle is equal to the velocity of the material particle. The particle is most likely to be found at a point near the peak of the wave-packet.

Example 19.1. Find the phase velocity and group velocity of an electron whose de Broglie wavelength is 1.2 Å.

Solution : Given : Wavelength $\lambda = 1.2$ Å $= 1.2 \times 10^{-10}$ m

From the de Broglie relation $\lambda = \frac{h}{mv}$, we have

Particle velocity $v = \frac{h}{m\lambda} = \frac{6.6\times10^{-34}}{9.1\times10^{-31}\times1.2\times10^{-10}}$

$$= 6.04\times10^{6}\ \text{ms}^{-1}$$

(*i*) Since, group velocity is equal to the particle velocity

Therefore, $v_g = 6.04\times10^{6}\ \text{ms}^{-1}$

(*ii*) Phase velocity $v_p = \frac{\omega}{k} = \frac{\hbar\omega}{\hbar k} = \frac{E}{p} = \frac{\frac{1}{2}mv^2}{mv} = \frac{v}{2}$

$$= \frac{6.04\times10^{6}}{2} = 3.02\times10^{6}\ \text{ms}^{-1}$$

Hence, phase velocity $v_p = 3.02\times10^{6}\ \text{ms}^{-1}$