Electromagnetic Wave
Lecture 1

Manoj Kr. Das

Associate Professor

Physics Department

J N College, Boko

Maxwell Equation:

On the basis of experimental result Maxwell put four compact mathematical equations lead to the discovery of electromagnetic wave and displacement current.

In case of static field the electric field can exist in absence of magnetic field and similarly magnetic field can exist in absence of electric field. But if field is time vrying then electric field can not exist without magnetic field and vice-versa.

Maxwell's equations for static field or time invariant case are

(a) Gauss's law in electrostatic

$$\nabla . D = \rho \quad or \quad \oiint D . ds = Q$$

First law says that the divergence of electric flux density at any point in static electric field is equal to the volume charge density at that point.

(b) Faraday’s law in electric field

$$\nabla \times E = 0 \quad OR \quad \oint E.dl$$

Second law says that curl of electric field intensity at any point in static electric field is zero. This implies that the static electric field is a conservative field or electric field intensity is irrotational.

(c) Ampere’s law in magnetostatic

$$\nabla \times H = J \quad or \quad \oint H.dl = i$$

Third law says that curl of the magnetic field intensity at any point in the static magnetic field is equal to the current density at that point.

(d) Gauss's law in magnetostatic

$$\nabla . B = 0 \quad or \quad \oiint B.ds = 0$$

Fourth law says that the divergence of magnetic flux density at any point in static magnetic field is equal to zero. This implies that the static magnetic flux density is solenoidal.

Here $D = \epsilon E, \ \ \epsilon = \epsilon_0 \epsilon_r \ , B = \mu H, \quad \mu = \mu_0 \mu_r, \quad J = \sigma E,$
$Q = \oiiint \rho dv, \quad i = \oiint jds$

Maxwell’s equation for time varying fields (Non Static Fields) can be derived as

When electric field and magnetic field vary with time then two Maxwell’s equation for static field remain same (Gauss’s law in electrostatic and magnetostatic), and remaining other two undergo changes.

(a) Faraday’s law of electromagnetic induction is

$$\nabla \times E = -\frac{\partial B}{\partial t} \;\; or \;\; \oint E dl = -\oiint \left(\frac{\partial B}{\partial t}\right) ds$$

(b)Modified Amper's law is

$$\nabla \times H = J + \frac{\partial D}{\partial t} \quad or \quad \oint H\, dl = j + \oiint \left(\frac{\partial D}{\partial t}\right)$$

(c) Gauss's law in electrostatic is

$$\nabla . D = \rho_v \quad or \quad \oiint D\, ds = Q$$

(d) Gauss's law in magnetostatic is

$$\nabla . B = 0 \quad or \quad \oiint B\, ds = 0$$

Wave Equation:

Let us consider a uniform linear medium having permeability ϵ and permeability μ and conductivity σ ; but not any charge or current other than that the determined by Ohm's law. Then

$$D = \epsilon E, \qquad B = \mu H, \qquad J = \sigma E, \qquad \rho = 0$$

So the Maxwell equations are

$$divD = \rho$$

$$divB = 0$$

$$curlE = -\frac{\partial B}{\partial t}$$

$$curlH = J + \frac{\partial D}{\partial t}$$

These equations take the form as

$$divE = 0$$

$$divH = 0$$

$$curlE = -\mu \frac{\partial H}{\partial t}$$

$$curlH = \sigma E + \epsilon \frac{\partial E}{\partial t}$$

Now taking $curl$ of the 3rd equation

$$curl\ curlE = -\mu \frac{\partial}{\partial t}(curlH)$$

Substituting the value of $curlH$ we get

$$curl\ curlE = -\mu \frac{\partial}{\partial t}\left(\sigma E + \epsilon \frac{\partial E}{\partial t}\right)$$

$$curl\ curlE = -\sigma\mu \frac{\partial E}{\partial t} - \epsilon\mu \frac{\partial^2 E}{\partial t^2} \rightarrow (i)$$

Similarly we get

$$curl\ curlH = -\sigma\mu \frac{\partial H}{\partial t} - \epsilon\mu \frac{\partial^2 H}{\partial t^2} \rightarrow (ii)$$

Now using vector identity

$$curl\ curlA = grad\ divA - \nabla^2 A \rightarrow (iii)$$

Now since $divE = 0 \ \ and \ \ divH = 0$, then from $equation \ (i)$ and $equation \ (ii)$ becomes

$$\nabla^2 E - \sigma\mu \frac{\partial E}{\partial t} - \epsilon\mu \frac{\partial^2 E}{\partial t^2} = 0 \ \rightarrow (iv)$$

$$\nabla^2 H - \sigma\mu \frac{\partial H}{\partial t} - \epsilon\mu \frac{\partial^2 H}{\partial t^2} = 0 \ \rightarrow (v)$$

These two equations represent wave equation which govern the electromagnetic field in homogenous linear medium in which the charge density is zero whether the medium is conducting or non conducting. Now the problem is that $equation \ (iv)$ and $equation \ (v)$ in such a manner that Maxwell’s equation are also satisfied.

It is more convenient to use the method of complex variable analysis for solution of wave equation. The time dependence of the field is taken to be $e^{i\omega t}$, so that

$$E(r,t) = E_s(r)e^{-i\omega t} \rightarrow (vi)$$

It may be noted that the physical electric field is obtained by taking the real part of $equation\ (vi)$. Furthermore $E_r(r)$ is in general complex, so that the actual electric field is proportional to $\cos(\omega t + \varphi)$, where φ is phase of $E_r(r)$. Using this $equation\ (vi)$, dropping common factor $e^{i\omega t}$, the $equation\ (v)$ gives as

$$\nabla^2 E_s + \omega^2 \epsilon\mu E_s + i\omega\sigma\mu E_s = 0 \ \rightarrow (vii)$$

Here the spatial electric field E_s depends on space coordinate i.e.

$$E_s = E_0 e^{ik.r}$$

Where k is propagation wave vector defined as

$$k = \frac{2\pi}{\lambda}\hat{n} = \frac{\omega}{v}\ \hat{n}$$

Where $\hat{n}$ being the unit vector along the k . Here r is the position vector from origin, v is the velocity of wave. So $equation\ (vi)$ may be written as

$$E(r,t) = E_0 e^{ik.r - i\omega t}\ \rightarrow (viii)$$

Here $E_0$is the complex amplitude and is constant in space and time. It is important to note that when field vector is in form of $equation\ (viii)$ i.e. operation of $grad, div, curl$ on field vector is equivalent to $grad \rightarrow ik, div = \nabla \rightarrow ik, curl = \nabla \times \rightarrow ik \times\ and\ \frac{\partial}{\partial t} \rightarrow i\omega$.

Plan Electromagnetic Equation in Free Space:

We know Maxwell's equations are

$$divD = 0$$

$$divB = 0$$

$$curl\ E = -\frac{\partial B}{\partial t}$$

$$curlH = J\frac{\partial E}{\partial t}$$

Here $B = \mu H$, $D = \epsilon H$, $J = \sigma E$

Now free space is characterized by

$$\rho = 0 , \sigma = 0 , \mu = \mu_0 , \epsilon = \epsilon_0$$

So Maxwell's equation's is reduced to

$$divE = 0$$

$$divH = 0$$

$$curlE = -\mu_0 \frac{\partial H}{\partial t}$$

$$curlH = \epsilon_0 \frac{\partial E}{\partial t}$$

Now taking $curl$ of $curlE$, we get

$$curl\, curlE = -\mu_0 \frac{\partial}{\partial t}(curlH)$$

$$curl\ curlE = -\mu_0 \frac{\partial}{\partial t}\left(\epsilon_0 \frac{\partial E}{\partial t}\right)$$

$$curl\ curlE = -\mu_0\epsilon_0 \frac{\partial^2 E}{\partial t^2} \rightarrow (i)$$

Now using $curl\ curlE = grad.divE - \nabla^2 E$.

Therefore $curl\ curlE = -\nabla^2 E$, $since\ \ divE = 0$

Putting this substitution, $equation\ (i)$ becomes

$$\nabla^2 E - \mu_0\epsilon_0 \frac{\partial^2 E}{\partial t^2} = 0 \rightarrow (ii)$$

Similarly we can get

$$curl\ curlH = \epsilon_0 \frac{\partial}{\partial t}(curlE)$$

$$curl\ curlH = \epsilon_0 \frac{\partial}{\partial t}\left(-\mu_0 \frac{\partial H}{\partial t}\right)$$

$$curl\ curlH = -\mu_0\epsilon_0 \frac{\partial^2 H}{\partial t^2} \ \rightarrow (iii)$$

Again using identity $curl\ curlH = grad\ divH - \nabla^2 H$, and $divH = 0$, we get $curl\ curlH = -\nabla^2 H$, so substitute this, $equation\ (iv)$ becomes

$$\nabla^2 H - \mu_0\epsilon_0 \frac{\partial^2 H}{\partial t^2} = 0 \ \rightarrow (iv)$$

Here $Equation\ (ii)$ and $equation\ (iv)$ represent wave equation governing electromagnetic field E and H in free space

These $equation\ (ii)$ and $equation\ (iv)$ are vector equations of same identical form which means E and H separately satisfies the same scalar wave equation of the form

$$\nabla^2 U - \mu_0 \epsilon_0 \frac{\partial^2 U}{\partial t^2} = 0 \ \rightarrow (v)$$

Where U is a scalar and can stand for one of the component of E and H. It is obvious that $equation\ (v)$ resembles with general wave equation

$$\nabla^2 U = \frac{1}{V} . \frac{\partial^2 U}{\partial t^2} \ \rightarrow (vi)$$

Where V is the velocity of wave. Comparing $equation\ (v)$ and $equation\ (vi)$, we see that the field vectors E and H are propagated in free space at a speed equal to

$$V = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Since $\mu_0 = 4\pi \times 10^{-7} \quad and\ \epsilon_0 = 8.85 \times 10^{-12}$

Therefore

$$V = 3 \times 10^8 \ {}^{mt}/_{sec} = C$$

Therefore it is a reasonable to write C the velocity of light in place of $\frac{1}{\sqrt{\mu_0 \epsilon_0}}$ as

$$\nabla^2 E - \frac{1}{C^2} \frac{\partial^2 E}{\partial t^2} = 0 \quad \rightarrow (vii)$$

$$\nabla^2 H - \frac{1}{C^2} \frac{\partial^2 H}{\partial t^2} = 0 \quad \rightarrow (viii)$$

$$\nabla^2 U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = 0 \quad \rightarrow (ix)$$

Solution of these equations are

$$E(r,t) = E_0 e^{ik.r-i\omega t} \quad \rightarrow (x)$$

$$H(r,t) = H_0 e^{ik.r-i\omega t} \quad \rightarrow (xi)$$

$$U(r,t) = U_0 e^{ik.r-i\omega t} \quad \rightarrow (xii)$$

Where $k = ku = \frac{2\pi}{\lambda} r = \frac{2\pi v}{c} r = \frac{\omega}{c} n$

And $\nabla E = ikE \;\; and \; \nabla H = ikH$ demand that $kE = 0$ and $kH = 0$

This means that electromagnetic field vector E and H are both perpendicular to the direction of propagation k . This implies that the electromagnetic waves are transverse in character.
Again

$$curlE = -\mu_0 \frac{\partial H}{\partial t}$$

$$ik \times E = -\mu_0(-\omega H)$$

$$k \times E = \mu_0 \omega H \ \rightarrow (xiii)$$

From $equation\ (xiii)$ it is obvious that H is perpendicular to k and E.

$$curlH = -\epsilon_0 \frac{\partial E}{\partial t} \rightarrow (xiv)$$

$$ik \times H = -\epsilon_0(-i\omega E) \rightarrow (xv)$$

$$k \times H = \epsilon_0 \omega E \rightarrow (xvi)$$

Again from $equation$ (xvi) it is seen that E is perpendicular k and H . This means E and H are mutually perpendicular and also they are perpendicular to the direction of propagation of wave.