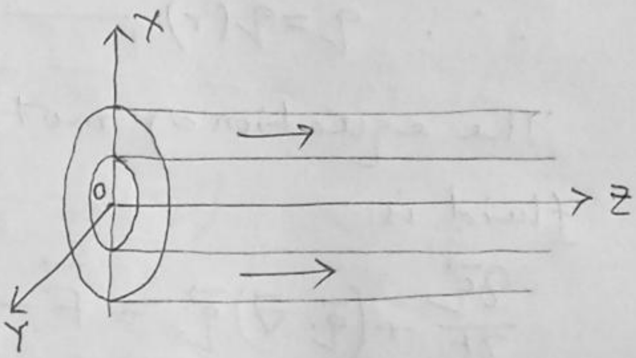
2012

Ex. A viscous liquid flows steadily parallel to the axis in the annular space between two coaxial cylinders of radii a and na $(n>1)$, show that the rate of discharge is

$$\frac{\pi P a^4}{8\mu}\left\{n^4-1-\frac{(n^2-1)^2}{\log n}\right\}$$

where P is the pressure gradient and μ is the coefficient of viscosity.



Sol^n Take z-axis along the common axis of the cylinders and x-axis and y-axis as shown in the figure.

Let $\vec{q}=(q_r, q_\theta, q_z)$ be the fluid velocity at the point $P(r,\theta,z)$ in the fluid.

Since the flow is parallel to z-axis

$$\therefore \left.\begin{aligned} q_r &= 0, \\ q_\theta &= 0 \end{aligned}\right\}$$

and let $q_z = q$

$\therefore$ The equation of continuity is

$$\vec{\nabla}\cdot\vec{q}=0$$

$$\Rightarrow \frac{1}{1.r.1}\left[\frac{\partial}{\partial r}(r.1.q_r)+\frac{\partial}{\partial\theta}(1.1.q_\theta)+\frac{\partial}{\partial z}(1.r.q_z)\right]=0$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial z}(rq_z)=0$$

$$\Rightarrow \frac{\partial q_z}{\partial z}=0 \quad \Rightarrow \frac{\partial q}{\partial z}=0 .$$

$\therefore$ q is independent of z.

Due to actual symmetric,

$\therefore$ q is independent of θ.

Since the motion is steady

$\therefore$ q is independent of 't'.

$\therefore$ $q = q(r) \longrightarrow (1)$

The equation of motion for an incompressible viscous fluid is

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\vec{\nabla})\vec{q} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p - \nu\vec{\nabla}\times\vec{\xi}$$

$$\Rightarrow \frac{\partial \vec{q}}{\partial t} + \frac{1}{2}\vec{\nabla}q^2 - \vec{q}\times\vec{\xi} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p - \nu\vec{\nabla}\times\vec{\xi} \longrightarrow (2)$$

Since the motion is steady and there is no body force acting on the fluid.

$$\left.\begin{aligned}\frac{\partial \vec{q}}{\partial t} &= 0\\ \vec{F} &= 0\end{aligned}\right\} \longrightarrow (3)$$

By virtue of (3) the equation (2) reduces to

$$\frac{1}{2}\vec{\nabla}q^2 - \vec{q}\times\vec{\xi} = -\frac{1}{\rho}\vec{\nabla}p - \nu\vec{\nabla}\times\vec{\xi} \longrightarrow (A)$$

$$\vec{\nabla}q^2 = \hat{r}\frac{\partial q^2}{\partial r} = \hat{r}\frac{dq^2}{dr} = \hat{r}\cdot 2qq' = 2qq'\hat{r} \longrightarrow (4)$$

$$\vec{\xi} = \vec{\nabla}\times\vec{q} = \frac{1}{1\cdot r\cdot 1}\begin{vmatrix}\hat{r} & r\hat{\theta} & \hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z}\\ 0 & 0 & q\end{vmatrix} = -\hat{\theta}q'$$

$$\vec{\nabla}\times\vec{\xi} = \frac{1}{1.r.1}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & -rq' & 0 \end{vmatrix}$$

$$= -\frac{\hat{z}}{r}\frac{d}{dr}(rq') \longrightarrow (5)$$

$$\vec{q}\times\vec{\xi} = q\hat{z}\times(-q'\hat{\theta})$$

$$= qq'(\hat{\theta}\times\hat{z}) = qq'\hat{r} \longrightarrow (6)$$

Using (4), (5) & (6) in (A), we get,

$$qq'\hat{r} - qq'\hat{r} = -\frac{1}{\rho}\vec{\nabla}p + \nu\frac{\hat{z}}{r}\frac{d}{dr}(rq')$$

$$\Rightarrow \vec{\nabla}p = \mu\frac{\hat{z}}{r}\frac{d}{dr}(rq')$$

$$\Rightarrow \hat{r}\frac{\partial p}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial p}{\partial \theta} + \frac{\hat{z}}{1}\frac{\partial p}{\partial z} = \frac{\mu}{r}\frac{d}{dr}(rq')\hat{z} \longrightarrow (7)$$

By equating the coefficient of $\hat{r}$, $\hat{\theta}$ and $\hat{z}$ in the equation (7) we derive,

$$\frac{\partial p}{\partial r} = 0 \longrightarrow (8.1)$$

$$\frac{\partial p}{\partial \theta} = 0 \longrightarrow (8.2)$$

$$\frac{\partial p}{\partial z} = \frac{\mu}{r}\frac{d}{dr}(rq') \longrightarrow (8.3)$$

(8.1) and (8.2) shows that p is independent of r and θ.

Further as the motion is steady

p is independent of t

$$\therefore p = p(z) \longrightarrow (9)$$

By the use of the equation (9), the equation (8.3) reduces to

$$\frac{dp}{dz} = \frac{\mu}{r}\frac{d}{dr}(rq')$$

$\Rightarrow$ a function of z = a function of r

$$\frac{\mu}{r}\frac{d}{dr}(rq') = \frac{dp}{dz} = \text{a constant} = -P \text{ (say)}$$

$$\Rightarrow \frac{d}{dr}(rq') = -\frac{Pr}{\mu}$$

$$\Rightarrow rq' = -\frac{Pr^2}{2\mu} + A$$

$$\Rightarrow \frac{dq}{dr} = -\frac{Pr}{2\mu} + \frac{A}{r}$$

$$\Rightarrow q = -\frac{Pr^2}{4\mu} + A\log r + B \longrightarrow (10)$$

where A and B are two arbitrary constants to be determined from the flow problem under considerations. The boundary condition of the flow problem are

$$q = 0 \text{ at } r = a \longrightarrow (11.1)$$

$$q = 0 \text{ at } r = na \longrightarrow (11.2)$$

Subjecting (11.1) and (11.2) in (10), we get,

$$0 = -\frac{Pa^2}{4\mu} + A\log a + B \longrightarrow (12.1)$$

$$0 = -\frac{Pn^2a^2}{4\mu} + A\log na + B \longrightarrow (12.2)$$

$$\therefore (12.2) - (12.1) \Rightarrow \frac{Pa^2}{4\mu} - \frac{Pn^2a^2}{4\mu} + A\log n = 0$$

$$\Rightarrow A\log n = \frac{Pa^2}{4\mu}(n^2-1)$$

$$\Rightarrow A = \frac{Pa^2(n^2-1)}{4\mu\log n}$$

$$(12\cdot 1) \Rightarrow B = \frac{Pa^2}{4\mu} - A\log a$$

$$= \frac{Pa^2}{4\mu} - \frac{Pa^2(n^2-1)}{4\mu \log n}\log a$$

$$\Rightarrow B = \frac{Pa^2}{4\mu}\left[1-(n^2-1)\frac{\log a}{\log n}\right]$$

Let Q be the rate of discharge (rate of flow or flux)

$$\therefore Q = \int_{\theta=0}^{2\pi}\int_{r=a}^{na} q\, r\, d\theta\, dr$$

$$= \int_{\theta=0}^{2\pi} d\theta \int_{r=a}^{na}\left[-\frac{Pr^2}{4\mu} + A\log r + B\right] r\,dr$$

$$= 2\pi\left[-\frac{P}{4\mu}\int_a^{na} r^3 dr + A\int_a^{na} r\log r\, dr + B\int_a^{na} r\,dr\right]$$

$$= 2\pi\left[A I_1 + B I_2 - \frac{P}{4\mu} I_3\right] \longrightarrow (13)$$

Now, $I_1 = \int_a^{na} \log r \cdot r\,dr$

$$= \log r \cdot \left[\frac{r^2}{2}\right]_a^{na} - \int_a^{na} \frac{1}{r}\cdot\frac{r^2}{2}\,dr$$

$$= \frac{1}{2}\left[n^2a^2\log na - a^2\log a\right] - \frac{1}{2}\int_a^{na} r\,dr$$

$$= \frac{1}{2}\left[n^2a^2\log na - a^2\log a\right] - \frac{1}{4}(n^2-1)a^2$$

$$= \frac{a^2}{2}(n^2\log na - \log a) - \frac{1}{4}(n^2-1)a^2$$

$$I_2 = \int_a^{na} r\,dr = \frac{1}{2}\left(n^2a^2 - a^2\right) = \frac{1}{2}\left(n^2-1\right)a^2$$

$$I_3 = \int_a^{na} r^3\,dr = \frac{1}{4}\left(n^4-1\right)a^4.$$

$$\therefore\ Q = 2\pi\left[T_1 + T_2 - T_3\right]$$

$$T_1 = A I_1 = \frac{Pa^2(n^2-1)}{4\mu \log n}\left[\frac{a^2}{2}\left(n^2\log na - \log a\right) - \frac{1}{4}(n^2-1)a^2\right]$$

$$= \frac{Pa^4(n^2-1)}{16\mu \log n}\left[2\left(n^2\log na - \log a\right) - (n^2-1)\right]$$

$$T_2 = B I_2 = \frac{Pa^2}{4\mu}\left[1 - \frac{n^2-1}{\log n}\log a\right]\frac{1}{2}(n^2-1)a^2$$

$$= \frac{Pa^4}{8\mu}\left[n^2-1 - \frac{(n^2-1)^2}{\log n}\log a\right]$$

$$T_3 = \frac{P}{4\mu} I_3 = \frac{P}{4\mu}\cdot\frac{1}{4}(n^4-1)a^4$$

$$= \frac{Pa^4}{16\mu}\left(n^4-1\right)$$

$$\therefore\ Q = 2\pi\cdot\frac{Pa^4}{16\mu}\left[\frac{n^2-1}{\log n}\left\{2\left(n^2\log na - \log a\right) - (n^2-1)\right\} + 2\left\{(n^2-1) - \frac{(n^2-1)^2\log a}{\log n}\right\} - (n^4-1)\right]$$

$$= 2\pi\cdot\frac{Pa^4}{16\mu}\left[\frac{(n^2-1)}{\log n}\left\{2\left(n^2\log na - \log a\right) - (n^2-1)\right\} + 2(n^2-1) - \frac{2(n^2-1)^2\log a}{\log n}\,\Big\} - (n^4-1)\right]$$

(85)

$$= \frac{\pi P a^4}{8\mu}\left[\frac{2(n^2-1)(n^2\log na-\log a)-(n^2-1)^2-2(n^2-1)^2\log a}{\log n}+\frac{2(n^2-1)-(n^4-1)}{1}\right]$$

$$= \frac{\pi P a^4}{8\mu}\left[\frac{(n^2-1)\{2n^2\log na-2\log a+n^2+1-2(n^2-1)\log a\}}{\log n}+2(n^2-1)-(n^4-1)\right]$$

$$= \frac{\pi P a^4}{8\mu}\left[\frac{(n^2-1)\{2n^2\log n-n^4+1\}}{\log n}+2(n^2-1)-(n^4-1)\right]$$

$$= \frac{\pi P a^4}{8\mu}\left[(n^2-1)\left(2n^2-\frac{n^2-1}{\log n}\right)+2(n^2-1)-(n^4-1)\right]$$

$$= \frac{\pi P a^4}{8\mu}\left[2n^2-\frac{(n^2-1)^2}{\log n}+2n^4+2n^2-2-n^4+1\right]$$

$$= \frac{\pi P a^4}{8\mu}\left[n^4-1-\frac{(n^2-1)^2}{\log n}\right]$$

#

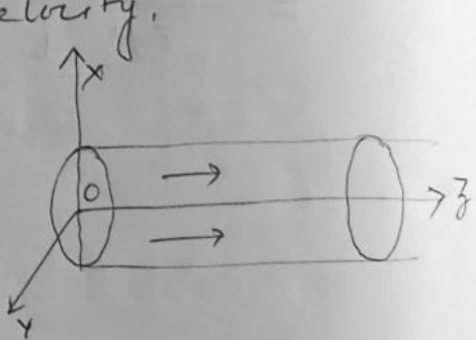
Ex In a steady motion of a viscous liquid through a cylindrical pipe, show that the equation of motion becomes

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=-\frac{P}{\mu},$$

where μ is the viscosity, w, $-P$ are respectively the velocity and pressure gradient along the axis of the cylinder. Further show that in the case of an elliptic cylinder across section given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the flux of the liquid through the cylinder if

$$Q=\frac{\pi}{4\mu}\frac{Pa^3b^3}{a^2+b^2}$$

Also abtain the maximum velocity.



Soln: Take z-axis along the axis of the pipe and other axes as shown in the figure.

Let $\vec{q}=(u,v,w)$ be the fluid velocity at the point $P(x,y,z)$ in the fluid,

In the present case since the fluid motion is parallel to z-axis

$\therefore u=0,\ v=0$

$\therefore \vec{q}=w\hat{k}$

The equation of continuity for an incompressible fluid is $\vec{\nabla}.\vec{q} = 0$

$$\Rightarrow \frac{\partial w}{\partial z} = 0$$

$\therefore$ w is independent of z.

Again, since the motion is steady

$\therefore$ w is independent of t.

and hence $w = w(x, y) \longrightarrow (1)$

The equation of motion for an incompressible viscous fluid is

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}.\vec{\nabla})\vec{q} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p + \nu \nabla^2 \vec{q} \longrightarrow (2)$$

Since the motion is steady and there is no body force acting on the fluid

$$\left.\begin{aligned} \frac{\partial \vec{q}}{\partial t} &= 0 \\ \vec{F} &= 0 \end{aligned}\right\} \longrightarrow (3)$$

By the use of (3), the equation (2) becomes

$$(\vec{q}.\vec{\nabla})\vec{q} = -\frac{1}{\rho}\vec{\nabla}p + \nu \nabla^2 \vec{q}$$

$$\Rightarrow \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right) w\hat{k} = -\frac{1}{\rho}\left[\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}\right)$$

$$+ \nu\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] w\hat{k}$$

$$\Rightarrow \vec{0} = -\frac{1}{\rho}\left(\hat{i}\frac{\partial p}{\partial x} + \hat{j}\frac{\partial p}{\partial y} + \hat{k}\frac{\partial p}{\partial z}\right) + \nu\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] w\hat{k}$$

$$\Rightarrow \hat{i}\frac{\partial p}{\partial x}+\hat{j}\frac{\partial p}{\partial y}+\hat{k}\frac{\partial p}{\partial z}=\mu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)w\hat{k} \longrightarrow (4)$$

By equating the coefficient of $\hat{i}, \hat{k}, \hat{k}$ in the equation (4)

We derive the following differential equation

$$\frac{\partial p}{\partial x}=0 \longrightarrow (5.1)$$

$$\frac{\partial p}{\partial y}=0 \longrightarrow (5.2)$$

$$\frac{\partial p}{\partial z}=\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) \longrightarrow (5.3).$$

equations (5.1) and (5.2) shows that p is independent of x and y.

Further the motion is steady

$\therefore$ p is independent of t

$\therefore$ $p=p(z) \longrightarrow (6)$

By virtue of (6) the equation (5.3) becomes

$$\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=\frac{dp}{dz}$$

$\Rightarrow$ a function of x and y = a function of z.

Hence we must have

$$\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=\frac{dp}{dz}= \text{a constant} = -P \text{ (say)}$$

$$\Rightarrow \frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=-\frac{P}{\mu}. \longrightarrow (7)$$

2nd part:

The equation of the cylinder is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The boundary condition for the flow problem is

$$w=0, \text{ for } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

i.e. $w=0$ for $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0 \longrightarrow (8)$

In order to satisfy the condition (8), we take W as follows,

$$w = A\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right) \longrightarrow (9)$$

$\therefore (9) \Rightarrow \frac{\partial w}{\partial x} = A\left(-\frac{2x}{a^2}\right)$

$$\frac{\partial^2 w}{\partial x^2} = -\frac{2A}{a^2}$$

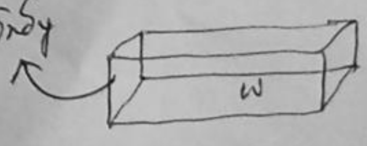
$\|^{ly}$, $\frac{\partial^2 w}{\partial y^2} = -\frac{2A}{b^2}$

$\therefore (7) \Rightarrow 2A\left(\frac{1}{a^2}+\frac{1}{b^2}\right) = \frac{P}{\mu}$

$\Rightarrow 2A \cdot \frac{a^2+b^2}{a^2b^2} = \frac{P}{\mu}$

$\Rightarrow A = \frac{Pa^2b^2}{2\mu(a^2+b^2)}$

Let Q be the flux across any normal section of the cylinder



$\therefore Q = \iint_R w\,dxdy \rightarrow (10)$, where $R=\left\{(x,y) \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\right\}$

$$\therefore \text{(10)} \Rightarrow Q = \iint_R A\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dxdy$$

$$= A\iint_R \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dxdy \longrightarrow \text{(11)}$$

Consider the transformation $x = ar\cos\theta,\ y = br\sin\theta$.

$$\therefore J = abr$$

$$\therefore \text{(11)} \Rightarrow Q = A\int_{\theta=0}^{2\pi}\int_{r=0}^{1}(1-r^2)\,abr\,drd\theta$$

$$= Aab\cdot 2\pi\int_{r=0}^{1}(r-r^3)dr$$

$$= 2Aab\pi\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1$$

$$= 2\pi ab\,A\left(\frac{1}{2}-\frac{1}{4}\right)$$

$$= \frac{\pi ab A}{2}$$

$$= \frac{\pi ab}{2}\cdot\frac{Pa^2b^2}{2\mu(a^2+b^2)}$$

$$\Rightarrow Q = \frac{P\pi a^3b^3}{4\mu(a^2+b^2)}.$$

3rd part:

We have $w = A\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$

$$= A\left[1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\right] \longrightarrow (12)$$

From (12) it is clear that w is maximum when

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$

i.e. when $x = 0$, $y = 0$.

And the maximum velocity is given by

$$w_{max} = A$$

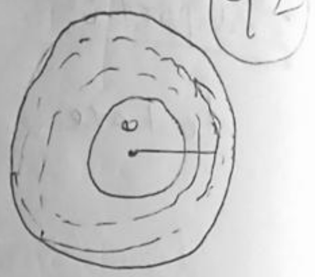
$$= \frac{Pa^2b^2}{2\mu(a^2+b^2)}.$$

#

13
Ex. A Liquid occupying the space between two co-axial circular cylinders is acted upon by a force $\frac{c}{r}$, per unit mass, where r is the distance from the axis, the line of force being circle around the axis. Prove that in the steady motion, the velocity at any point is given by the formula

$$\frac{c}{2\nu}\left\{\frac{b^2}{r}\,\frac{r^2-a^2}{b^2-a^2}\log\frac{b}{a} - r\log\frac{r}{a}\right\}$$

where a and b are two radii and ν is the kinematic viscocity.



Sol^n Take the z-axis along the common axis of the two cylinders.

Let $\vec{q} = (q_r, q_\theta, q_z)$ be the fluid velocity at the point $P(r, \theta, z)$ in the fluid.

In the present case, since the lines of force are clearly round the z-axis, therefore $q_r = 0$, $q_z = 0$

and let $q_\theta = q$

$\therefore \vec{q} = q_\theta \hat{\theta} = q\hat{\theta}$

The equation of continuity for an incompressible fluid is

$\vec{\nabla}.\vec{q} = 0$

$\Rightarrow \frac{1}{1.r.1}\left[\frac{\partial}{\partial r}(r.1.q_r) + \frac{\partial}{\partial \theta}(1.1.q_\theta) + \frac{\partial}{\partial z}(1.r.q_z)\right] = 0$

$\Rightarrow \frac{\partial q_\theta}{\partial \theta} = 0$

$\Rightarrow \frac{\partial q}{\partial \theta} = 0$

$\therefore q$ is independent of θ.

Again, since the motion in two dimensional with the z-~~axis~~ planes as the plane of the motion.

Hence q is independent of z

further, since the motion is steady

$\therefore q$ is independent of t

$\therefore q = q(\theta)$. $\longrightarrow$ (1)

Q3

The equation of motion for an incompressible viscous fluid is

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\vec{\nabla})\vec{q} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p - \nu\,\vec{\nabla}\times\vec{\xi}$$

$$\Rightarrow \frac{\partial \vec{q}}{\partial t} + \frac{1}{2}\vec{\nabla} q^2 - \vec{q}\times\vec{\xi} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p - \nu\,\vec{\nabla}\times\vec{\xi} \longrightarrow (A)$$

Since the motion is steady,

$$\frac{\partial \vec{q}}{\partial t} = 0 \longrightarrow (2)$$

∴ Force per unit mass

$$\vec{F} = \frac{c}{r}\hat{\theta} \longrightarrow (3)$$

$$\vec{\nabla} q^2 = \frac{\hat{r}}{1}\frac{\partial}{\partial r}(q^2) + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta}(q^2) + \frac{\hat{z}}{1}\frac{\partial}{\partial z}(q^2)$$

$$= \hat{r}\cdot 2qq' = 2qq'\hat{r} \longrightarrow (4)$$

$$\vec{\xi} = \vec{\nabla}\times\vec{\xi}$$

$$= \frac{1}{1.r.1}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & rq & 0 \end{vmatrix} = \frac{\hat{z}}{r}\frac{d}{dr}(r\vec{q}) = \xi\hat{z},$$

$$\xi = \frac{1}{r}\frac{d}{dr}(r\vec{q}).$$

$$\vec{\nabla}\times\vec{\xi} = \frac{1}{1.r.1}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & 0 & \xi \end{vmatrix} = -\hat{\theta}\frac{d\xi}{dr} = -\hat{\theta}\xi' \longrightarrow (5)$$

$$\therefore \vec{q}\times\vec{\xi} = q\hat{\theta}\times\xi\hat{z}$$

$$= q\xi\hat{r} \longrightarrow (6)$$

$$\therefore (A) \Rightarrow qq'\hat{r} - q\xi\hat{r} = \frac{c}{r}\hat{\theta} - \frac{1}{\rho}\vec{\nabla}p + \nu\,\xi'\hat{\theta}$$

$$\Rightarrow q q' \hat{r} - q \xi \hat{r} = -\frac{c}{r}\hat{\theta} - \frac{1}{\rho}\left[\hat{r}\frac{\partial p}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial p}{\partial \theta} + \hat{z}\frac{\partial p}{\partial z}\right] + \nu \xi' \hat{\theta} \rightarrow (7)$$

By equating the coefficient of $\hat{r}, \hat{\theta}, \hat{z}$ in the equation (7), we obtain the following differential equations.

$$q q' - q\xi = -\frac{1}{\rho}\frac{\partial p}{\partial r} \longrightarrow (8.1)$$

$$0 = \frac{c}{r} - \frac{1}{r\rho}\frac{\partial p}{\partial \theta} + \nu \xi' \longrightarrow (8.2)$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} \longrightarrow (8.3)$$

(8.3) shows that p is independent of z.

Due to symmetry p is independent of θ.

Since, the motion is steady

$\therefore$ p is independent of 't'

$$\therefore \quad p = p(r) \longrightarrow (9)$$

By virtue of (9), the equation (8.2) reduces to

$$\nu \xi' = -\frac{c}{r}$$

$$\Rightarrow \xi' = -\frac{c}{\nu r}$$

$$\Rightarrow d\xi = -\frac{c}{\nu r}\,dr$$

$$\Rightarrow \xi = -\frac{c}{\nu}\log r + A$$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}(rq) = A - \frac{c}{\nu}\log r$$

$$\Rightarrow \frac{d}{dr}(rq) = Ar - \frac{cr}{\nu}\log r$$

$$\Rightarrow rq = \frac{Ar^2}{2} - \frac{c}{\nu}\left[\log r \cdot \frac{r^2}{2} - \int \frac{1}{r}\cdot\frac{r^2}{2}\,dr\right] + B.$$

$$\Rightarrow rq = \frac{Ar^2}{2} - \frac{c}{\nu}\left[\frac{r^2}{2}\log r - \frac{r^2}{4}\right] + B$$

$$\Rightarrow q = \frac{Ar}{2} - \frac{cr}{2\nu}\left(\log r - \frac{1}{2}\right) + \frac{B}{r}$$

$$= \frac{Ar}{2} - \frac{cr}{2\nu}\log r + \frac{cr}{4\nu} + \frac{B}{r}$$

$$= Kr - \frac{cr}{2\nu}\log r + \frac{B}{r} \longrightarrow (1), \quad K = \frac{A}{2} + \frac{c}{4\nu}$$

The boundary condition for the slow problem are,

$$q = 0, \text{ at } r = a \longrightarrow (11.1)$$

$$q = 0, \text{ at } r = b \longrightarrow (11.2)$$

(5) can be written as

$$\frac{q}{r} = K - \frac{c}{2\nu}\log r + \frac{B}{r^2} \longrightarrow (12)$$

Subjecting (12) to the condition (11.1) and (11.2), we get,

$$0 = K - \frac{c}{2\nu}\log a + \frac{B}{a^2} \longrightarrow (13.1)$$

$$0 = K - \frac{c}{2\nu}\log b + \frac{B}{b^2} \longrightarrow (13.2)$$

$\therefore (13.1) - (13.2) \Rightarrow$

$$0 = \frac{c}{2\nu}\log\frac{b}{a} + B\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$$

$$\Rightarrow 0 = \frac{c}{2\nu}\log\frac{b}{a} + B\cdot\frac{b^2 - a^2}{a^2 b^2}$$

$$\Rightarrow B\cdot\frac{b^2 - a^2}{a^2 b^2} = -\frac{c}{2\nu}\log\frac{b}{a}$$

$$\Rightarrow B = -\frac{c\, a^2 b^2 \log\frac{b}{a}}{2\nu(b^2 - a^2)}$$

(12) — (13·1) $\Rightarrow$

$$\frac{q}{r} = \frac{c}{2\nu}\log\frac{a}{r} + B\left(\frac{1}{r^2} - \frac{1}{a^2}\right)$$

$$= \frac{c}{2\nu}\log\frac{a}{r} - \frac{c\,a^2 b^2 \log\frac{b}{a}}{2\nu\,(b^2-a^2)} \cdot \frac{a^2-r^2}{a^2 r^2}$$

$$= \frac{c}{2\nu}\left[\log\frac{a}{r} - \frac{b^2(a^2-r^2)}{r^2(b^2-a^2)}\log\frac{b}{a}\right]$$

$$= \frac{c}{2\nu}\left[\frac{(r^2-a^2)b^2 - \log\frac{b}{a}}{r^2(b^2-a^2)} - \log\frac{b}{a}\right]$$

$$\Rightarrow q = \frac{c}{2\nu}\left[\frac{(r^2-a^2)b^2 - \log\frac{b}{a}}{r(b^2-a^2)} - r\log\frac{b}{a}\right] \#$$

Ex. The space between two coaxial cylinders of radii a and b is filled with a viscous fluid and the cylinders are made to related with angular velocities ω_1 and ω_2. Prove that in the steady motion the angular velocity of the fluid is given by

$$\omega = \frac{a^2(b^2-r^2)\,\omega_1 + b^2(r^2-a^2)\,\omega_2}{r^2(b^2-a^2)}.$$

Soln: Proceeding as in the previous example and putting $c=0$, we get $\xi' = 0$

$$\Rightarrow \frac{d\xi}{dr} = 0$$

$$\Rightarrow \zeta = A = \text{a constant}$$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}(rq) = A$$

$$\Rightarrow \frac{d}{dr}(rq) = Ar$$

$$\Rightarrow rq = A\frac{r^2}{2} + B$$

$$\Rightarrow q = \frac{Ar}{2} + \frac{B}{r}$$

$$\Rightarrow r\omega = \frac{Ar}{2} + \frac{B}{r}$$

$$\Rightarrow \omega = \frac{A}{2} + \frac{B}{r^2}$$

$$\Rightarrow \omega = \lambda + \frac{B}{r^2} \longrightarrow (A), \quad \lambda = \frac{A}{2}$$

The boundary conditions for the flow problem are

$$\omega = \omega_1 \quad \text{at } r = a \longrightarrow (1.1)$$

$$\omega = \omega_2 \quad \text{at } r = b \longrightarrow (1.2)$$

subjecting (A) to the condition (1.1) & (1.2), we get,

$$\omega_1 = \lambda + \frac{B}{a^2} \longrightarrow (2.1)$$

$$\omega_2 = \lambda + \frac{B}{b^2} \longrightarrow (2.2)$$

$$(2.1) - (2.2) \Rightarrow \omega_1 - \omega_2 = B\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$$

$$= B.\frac{b^2 - a^2}{a^2 b^2}$$

$$\Rightarrow B = \frac{a^2 b^2(\omega_1 - \omega_2)}{b^2 - a^2}$$

$$(A) - (2.1) \Rightarrow \omega - \omega_1 = B\left(\frac{1}{r^2} - \frac{1}{a^2}\right)$$

$$= \frac{a^2 b^2(\omega_1 - \omega_2)}{(b^2 - a^2)}\left(\frac{a^2 - r^2}{a^2 r^2}\right)$$

$$\Rightarrow \omega = \omega_1 + \frac{b^2(a^2-r^2)}{r^2(b^2-a^2)}(\omega_1-\omega_2)$$

$$= \frac{\omega_1 r^2(b^2-a^2) + b^2(a^2-r^2)\omega_1 - b^2(a^2-r^2)\omega_2}{r^2(b^2-a^2)}$$

$$= \frac{\omega_1(b^2r^2 - a^2r^2 + b^2a^2 - b^2a^2) + b^2(r^2-a^2)\omega_2}{r^2(b^2-a^2)}$$

$$\Rightarrow \omega = \frac{\omega_1 a^2(b^2-r^2) + \omega_2(r^2-a^2)b^2}{r^2(b^2-a^2)}$$

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Ex. Two concentric infinite circular cylinders of radii a, b $(b > a)$ contains a viscous liquid in between the liquid is set in motion by relatively the other cylinder with a constant angular velocity Ω, write the inner cylinder is kept at rest. Discuss the motion and show that the couple per unit length on the inner cylinder due to the viscous drag has moment $\dfrac{4\pi\mu a^2 b^2 \Omega}{b^2-a^2}$.

Soln: Proceeding as pervious example to the state

$$\omega = \lambda + \frac{\beta}{r^2} \longrightarrow (A)$$

The boundary conditions for the present flow problem are

$$\omega = \Omega \quad \text{for } r = b \longrightarrow (i)$$

$$\omega = 0 \quad \text{for } r = a \longrightarrow (ii)$$

Subjecting Ⓐ to ① and ⑪, we get,

$$\Omega = \lambda + \frac{B}{b^2} \longrightarrow (2.1)$$

$$0 = \lambda + \frac{B}{a^2} \longrightarrow (2.2)$$

$\therefore$ $(2.1) - (2.2) \Rightarrow \Omega = B\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$

$$= B\,\frac{b^2-a^2}{a^2b^2}$$

$$\Rightarrow B = \frac{a^2b^2\Omega}{a^2-b^2}$$

Ⓐ $-(2.2) \Rightarrow$

$$\omega = B\left(\frac{1}{r^2} - \frac{1}{a^2}\right)$$

$$= \frac{a^2b^2\Omega}{a^2-b^2}\left(\frac{a^2-r^2}{a^2r^2}\right)$$

$$= \frac{b^2(a^2-r^2)\Omega}{r^2(a^2-b^2)}$$

The moment of couple present length of the inner cylinder due to viscous that is given by

$$G = \left(\tau_{r\theta}\,.\,2\pi r\,.\,1\right)r\Big]_{r=a}$$

$$= \left[2\pi r^2\tau_{r\theta}\right]_{r=a} \longrightarrow (3)$$

We have,

$$\tau_{r\theta} = \mu\left[\frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r}\right]$$

$$= \mu \left[\frac{\partial}{\partial r}(r\omega) - \frac{r\omega}{r} \right]$$

$$= \mu \left[r \frac{\partial \omega}{\partial r} + \omega - \omega \right]$$

$$= \mu r \frac{\partial \omega}{\partial r}$$

$$= \mu r . \frac{b^2 \Omega}{a^2 - b^2} \frac{d}{dr} \left(\frac{a^2 - r^2}{r^2} \right)$$

$$= \mu r . \frac{b^2 \Omega}{a^2 - b^2} \frac{d}{dr} \left(\frac{a^2}{r^2} - 1 \right)$$

$$= \frac{\mu r b^2 \Omega}{a^2 - b^2} \left[a^2 (-2) r^{-3} \right]$$

$$= \frac{\mu r b^2 \Omega}{b^2 - a^2} \frac{2a^2}{r^3}$$

$$\Rightarrow G = 2\pi r^2 \left. \frac{2a^2 b^2 \mu \Omega}{(b^2 - a^2) r^2} \right]_{r=a}$$

$$= \frac{4\pi a^2 b^2 \mu \Omega}{(b^2 - a^2)}$$

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