Problems on Theory of Equations :

1. If α, β be the roots of the equation $x^2 - 4x - 21 = 0$ then find the value of the following expressions.

(i) $\frac{\alpha}{3\beta} + \frac{\beta}{3\alpha}$

(ii) $\frac{1}{\alpha^2\beta} + \frac{1}{\beta^2\alpha}$

(iii) $(3\alpha - 1)(3\beta - 1)$

(iv) $\frac{\alpha + 3}{\beta} + \frac{\beta + 3}{\alpha}$

2. The roots of the equation $2x^2 - 7x + 5 = 0$ are α and β. find

(i) $\frac{1}{\alpha} + \frac{1}{\beta}$

(ii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

(iii) $\frac{\alpha + 2}{\beta + 2} + \frac{\beta + 2}{\alpha + 2}$

3. The roots of the equation $x^2 + 6x - 4 = 0$ are α, β. Find the quadratic equation whose roots are

(i) α^2 and β^2

(ii) $2/\alpha$ and $2/\beta$

(iii) $\alpha^2\beta$ and $\beta^2\alpha$

4. If α, β are the roots of $7x^2 + ax + 2 = 0$ and if $\beta - \alpha = -13/7$. Find the values of a.

5. If one root of the equation $2y^2 - ay + 64 = 0$ is twice the other then find the values of a.

6. If one root of the equation $3x^2 + kx + 81 = 0$ (having real roots) is the square of the other then find k.

7. Use Descartes' Rule of Signs to describe all possibilities for the number of positive, negative, and imaginary zeros of

(i) $p(x) = x^4+x^3+x^2+x+12$.

(ii) $q(x) = 2x^4-2x^3-5x^2-x+8$

8. Give the statement of the Descartes's rule of sign of a polynomial equation.

9. State the fundamental theorem of algebra regarding roots of polynomials.

10. Prove that Every polynomial of degree $n > 0$ has exactly n roots where the roots are counted according to their multiplicities.

11. Form the equation of the lowest degree with rational coefficients whose roots are $3 + \sqrt{5}$ and 1.

12. Solve the equation $x^4 + 4x^3 + 5x^2 + 2x - 2 = 0$ which one root is $-1 + i$.

13. If α, β, γ be the roots of the equation $x^2 + ax + b = 0$, find the value of

(i) $\sum\frac{\alpha}{\beta\gamma}$ (ii) $\sum\frac{\alpha\beta}{\gamma}$ (iii) $\sum\frac{\alpha}{\beta+\gamma}$ (iv) $\sum\frac{1}{\beta+\gamma}$ (v) $\sum\alpha^3$ (vi) $\sum\frac{\beta}{\gamma}+\frac{\gamma}{\beta}$

14. . If α, β, γ are the roots of the equation $ax^3 + bx^2 + cx + d = 0$ find values of

(i) $(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)$ (ii) $(\beta + \gamma - 2\alpha)(\gamma + \alpha - 2\beta)(\alpha + \beta - 2\gamma)$

15. If the roots of the equation $x^3 + px^2 + qx + r = 0$ are in A.P., show that $2p^3 - 9pq + 27r = 0$.

16. Solve the equation $x^3 - 7x^2 + 14x - 8 = 0$ given that its roots are in G.P.

17. If the sum of the two roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ equals the sum of the other two, prove that $p^3 + 8r = 4pq$.

18. Solve the equation $x^3 - 4x^2 - 3x + 18 = 0$ given that two of its roots are equal.

19. Prove that the sum of the cubes of the roots of $x^3-6x^2 + 11x -6 =0$ is 36

20. If A is symmetric or skew-symmetric, prove that (i) A^2 is symmetric (ii) $A A' = A' A$.

11. If A and B are both symmetric or skew-symmetric of the same order and AB = BA, then show that AB is symmetric.

12. Find all matrices which commute with the matrix $\begin{bmatrix} 7 & -3 \\ 5 & -2 \end{bmatrix}$.

13. Show that the matrix $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ is a solution of the equation $A^2 - 5A + 7I = 0$.

14. Define the inverse of a matrix. State a necessary condition for a matrix to be invertible.

15. Prove that for two invertible matrices A and B, (i) $((A)^{-1})^{-1} = A$ (ii) $(AB)^{-1} = B^{-1}A^{-1}$

16. When a matrix is called idempotent ? Show that if A is idempotent, then I – A is also an idempotent.

17. For a non-singular matrix A, show that $A(\text{Adj } A) = (\text{Adj } A)A = |A|\, I$

18. Solve : $x + y + z = 9$, $2x + 5y + 7z = 52$, $2x + y - z = 0$ by the Cramer's rule.

19. Define rank of a matrix. Find the ranks of the following matrices

(i) $\begin{bmatrix} 2 & 1 & 3 \\ 6 & 3 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 0 \\ 3 & 4 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 3 & 2 \end{bmatrix}$

20. Find the ranks of the following matrix

$$\begin{bmatrix} 2 & 0 & 4 & 6 \\ 3 & 2 & 1 & 5 \\ 7 & 2 & 9 & 17 \end{bmatrix}$$