Important practice question

1. Short Answer Questions ($1 \times 5 = 5$ marks)

(a) For $n > 1$, the sum of all distinct nth roots of unity is ________.

(b) If an algebraic equation is of degree n, which of the following statements is always true?

(i) It can have at most $n - 1$ roots
(ii) It has exactly n distinct real roots
(iii) It has n roots in total, real or complex
(iv) It can have infinitely many roots

(c) For a polynomial with real coefficients, the number of positive real roots is equal to the number of sign changes or less by a multiple of ________.

(d) State whether the following statement is true or false: For any two 2×2 matrices A and B, $(A + B)^2 = A^2 + 2AB + B^2$ holds always.

(e) A system of m linear equations in n unknowns is called inconsistent if it has:

(i) Exactly a single root satisfying all equations
(ii) No common root that satisfies every equation
(iii) Multiple roots satisfying all equations simultaneously
(iv) None of the above

2. Answer any six questions ($2 \times 6 = 12$ marks)

(a) Using Descartes' Rule of Signs, determine the possible number of real and complex roots of the equation $x^4 - 3x^3 + 3x^2 - x + 2 = 0$.

(b) If α and β are the roots of the equation $ax^2 - bx + c = 0$, express the sum and product of the roots in terms of the coefficients.

(c) Define symmetric functions of the roots of an equation. Give one example.

(d) If A is a square matrix such that $A^T = -A$, what special property does A have? Explain briefly .

(e) If a square matrix A satisfies $A^2 = 0$ (the zero matrix), can A^{-1} exist? Justify your answer.

(f) When does a homogeneous system of linear equations have only the trivial solution, and when does it have infinitely many solutions?

3. Section 3 ($5 \times 4 = 20$ marks)

(a) Let $n > 1$ be a positive integer. Find all real nth roots of 1 for the cases when n is even and when n is odd.

(b) If the roots of the quadratic equation $x^2 - 5x + 6 = 0$ are transformed such that each root is increased by 3, form the new quadratic equation whose roots are the transformed roots.

(c) Explain whether the sets of all symmetric and all skew-symmetric matrices are each closed under matrix addition.

(d) Find the reduced row echelon form of the following matrix and determine its rank:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 1 & 2 & 3 \end{pmatrix}$$

4. Long Answer Questions ($5 \times 7 = 35$ marks)

(a) For the cubic equation $x^3 - 4x^2 + px - q = 0$, two of its roots are 1 and $(p - 2)$. Using only the relations between roots and coefficients, find the values of p and q.

(b) Solve by Cardan's method: $x^3 - 6x^2 + 11x - 6 = 0$.

(c) If $AB = I$ for square matrices A and B of the same order, prove that $B = A^{-1}$.

(d) Prove that the following statements for a square matrix A are equivalent:

- A is nonsingular.
- The equation $Ax = 0$ has only the trivial solution.
- $\det(A) \neq 0$.

(e) Explain briefly how the augmented matrix of a linear system of equations helps in determining whether the system is consistent.

(f) Test the consistency of the following system of equations and solve if it is consistent:

$$\begin{aligned} x + y + z &= 6 \\ x + 3y + z &= 10 \\ x + 2y + 2z &= 8 \end{aligned}$$

(g) For a homogeneous system $Ax = 0$, explain how the number of free variables in the reduced matrix relates to the number of possible solutions.

(h) Determine all solutions of the following homogeneous system:

$$\begin{aligned} x + 2y - z &= 0 \\ 2x + 5y - 3z &= 0 \\ 3x + 8y - 5z &= 0 \end{aligned}$$