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14 (MAT–2) 2016 (N/O)

2023

MATHEMATICS

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Estd.-1964
Boko, Dist - Kamrup

Paper : MAT–2016

(New and Old Course)

(Complex Analysis)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any four*** parts : 5×4=20

   (a) Prove the Cauchy-Goursat theorem for any simple closed curve.

   (b) Let $f(z)$ be analytic in a simply-connected region R and let a and z be points in R. Prove that $F(z)=\int_a^z f(u)\,du$ is analytic in R and $F'(z)=f(z)$.

Contd.

(c) Let $f(z)$ be analytic inside and on a simple closed curve C and a is any point inside C. Prove that

$$f(a)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

(d) Let $f(z)$ be analytic inside and on a circle C of radius r and centre at $z=a$. Prove that

(i) $\left|f^{(n)}(a)\right| \le \frac{M.n!}{r^n}\,(n=0,1,2,\ldots)$,

where M is a constant such that $|f(z)| < M$ on C

(ii) $f(a)=\frac{1}{2\pi}\int_0^{2\pi} f\left(a+re^{i\theta}\right)d\theta$

3+2=5

(e) Prove that every polynomial equation

$P(z)=a_0+a_1z+a_2z^2+\ldots.+a_nz^n=0$,

where the degree $n \ge 1$ and $a_n \neq 0$, has exactly n roots.

2. Answer ***any two*** parts : 8×2=16

(a) (i) If $f(z)$ is analytic inside a circle C with centre at a, prove that for all z inside C,

$$f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{\lfloor 2}(z-a)^2+\ldots$$

5

(ii) If $a > e$, prove that the equation $az^n=e^z$ has n roots inside $|z|=1$. 3

(b) (i) Prove that all the roots of $z^7-5z^3+12=0$ lie between the circles $|z|=1$ and $|z|=2$. 4

(ii) If $f(z)=z^5-3iz^2+2z-1+i$, evaluate $\oint_C \frac{f'(z)}{f(z)}\,dz$, where C encloses all the zero of $f(z)$.

3

(iii) Using Laurent series expansion, classify the singularity of $e^{1/z}$ at $z=0$. 1

(c) (i) How can one classify poles, removable singularity and essential singularity for a single valued function with the help of its Laurent series expansion? Discuss briefly with proper examples. 6

(ii) Find the zeros and poles of

$$\left(\frac{z+1}{z^2+1}\right)^2.$$ 2

3. Find the zeros and poles of

$$f(z)=\frac{z^2+4}{z^3+2z^2+2z}$$ and determine the residues at the poles. 4

OR

Find the residues of the function

$$\frac{\cot \pi z}{(z-a)^2}$$

4. Answer ***any two*** parts : 6×2=12

(a) Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz$

around the circle C with equation $|z|=3$.

(b) Prove that $\int_0^\infty \frac{dx}{x^4+1}=\frac{\pi}{2\sqrt{2}}$.

(c) Show that $\int_0^\infty \frac{\cos mx}{x^2+1}dx=\frac{\pi}{2}e^{-m}, m>0$.

(d) Use residue theorem to prove that

$$\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta}=\frac{2\pi}{\sqrt{a^2-b^2}} \text{ if } a>|b|$$

(e) Prove that $\int_0^\infty \frac{\log(x^2+1)}{x^2+1}dx=\pi\log 2$

5. Answer ***any two*** parts : 8×2=16

(a) (i) Find all bilinear transformations which transforms the unit circle $|z| \leq 1$ into the unit circle $|w| \leq 1$. 6

(ii) Discuss the significance of the vanishing of the Jacobian for a bilinear transformation. 2

(b) (i) Find a bilinear transformation which maps the points, $1, i, -1$ in the z-plane into the points $0, 1, \infty$ in the w-plane. Show also that by this transformation the area of the circle $|z| = 1$ is represented in the w-plane by the half plane above the real axis. 6

(ii) Define a single valued function and a multiple valued function with the help of example. Mention a singularity which appears in multiple valued function but not in single valued function. 1+1=2

(c) (i) Define conformal mapping and isogonal mapping. Give proper examples of both. 4

(ii) The interior of the square S with vertices at $1, 2, 1+i, 2+i$ is mapped into a region S' by means of the transformation $w = \sin \pi z$. Using the theorem that connects analyticity and conformity of a function at a certain point, show that the interior angles of S' are right angles. 4

6. Answer ***any three*** parts : 4×3=12

(a) Let $F_1(z)$ be analytic in R_1 and on the boundary JKLM. Suppose that we can find a function $F_2(z)$ analytic in R_2 and on the boundary JKLM such that $F_1(z) = F_2(z)$ on JKLM. Prove that the function $F(z) = \begin{cases} F_1(z) \text{ for } z \in R_1 \\ F_2(z) \text{ for } z \in R_2 \end{cases}$ is analytic in the region R which is composed of R_1 and R_2.

(b) Given the identity $\sin^2 z + \cos^2 z = 1$ holds for all real z. Prove that it also holds for all complex z using the Schwarz's Reflection principle.

(c) Show that the infinite product $\prod_{n=1}^{\infty}\left(1+\frac{z}{n\pi}\right)e^{-\frac{z}{n\pi}}$ is absolutely convergent.

(d) Given the function $F_1(z)=\int_0^{\infty} t^3 e^{-zt}\,dt$. Find a function which is the analytic continuation of $F_1(z)$ into the left hand plane $Re(z)<0$.

(e) Show that the infinite product $e^{1+1}e^{-1-1/2}e^{1+1/3}e^{-1-1/4}\ldots$ oscillates between two finite values.
