14 (MAT–2) 2036 (N/O)

2023

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MATHEMATICS

Paper : MAT–2036

(New and Old Course)

(Functional Analysis)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any two*** parts : 5×2=10

   (a) Prove that a normed space X is a Banach space if and only if each absolutely convergent series in X converges.

   (b) Let X be a normed space. Prove that

   (i) if S is subspace of X, then $\overline{S}$ is a subspace of X;

   (ii) if C is a convex subset of X, then $\overline{C}$ and C^C are convex.

   2+3=5

Contd.

(c) Prove that every ball centred at origin of a normed space, whether open or closed is balanced and absorbing. 2+3=5

2. Answer ***any two*** parts : 5×2=10

(a) Suppose that X and Y are normed spaces and T is a bounded linear operator from X into Y.

Prove that

(i) if $x \in X$, then $\|Tx\| \leq \|T\| \|x\|$;

(ii) the number $\|T\|$ is the smallest non-negative real number M such that

$\|Tx\| \leq M \|x\|$, for every $x \in X$. 2+3=5

(b) (i) Let X and Y be normed spaces and let $T : X \to Y$ be a linear operator. Prove that if T is continuous at 0 then T is uniformly continuous on X.

(ii) Let X be the vector space of finitely non-zero sequences with the l_∞-norm. Construct an unbounded linear operator $T : X \to F$ with justification. 2+3=5

(c) Define normed space isomorphism. Let T be a linear operator from a normed space X into a normed space Y. Prove that the operator T is isomorphism if and only if there are positive constants s and t such that $s\|x\| \leq t\|Tx\| \leq t\|x\|$, whenever $x \in X$.

3. Answer ***any two*** parts : 5×2=10

(a) State and prove the closed graph theorem.

(b) Suppose T is a bounded linear operator from a Banach space X onto a Banach space Y. Prove that

$B_Y(0, r) \subset T(B_X(0, 1))$ for some $r > 0$.

(c) Let X and Y be normed spaces and $T : D(T) \to Y$ be a linear operator, where $D(T) \subset X$. Prove that T is closed if and only if it has the following property :

if $x_n \to x$ where $x_n \in D(T)$, and $T(x_n) \to y$, then $x \in D(T)$ and $Tx = y$.

(d) Applying the uniform boundedness theorem, show that the normed space X of all polynomials with norm defined by $\|x\| = max\,|a_j|$, where (a_j's are the coefficients of the polynomial x) is not complete.

4. Answer ***any two*** parts : 5×2=10

(a) Let M and N be closed subspaces of a normed space X. Prove that if either M or N is finite dimensional, then $M+N$ is a closed subspace of X.

(b) Let Y be a closed subspace of a normed space X. Suppose that $x \in X \to Y$. Show that there is a bounded linear functional f on X such that $\|f\| = 1$, $f(x) = d(x, y)$ and $y \subseteq ker\ f$.

(c) Let X be a vector space and p be a non-negative real valued positive homogeneous function on X. Let $A_p = \{x : x \in X,\ p(x) < 1\}$.

Show that the set A_p is absorbing, and p is the Minkowski functional of A_p. Moreover, if p is a sublinear functional, then show that A_p is a convex set.

5. Answer ***any two*** parts : 5×2=10

(a) Prove that the dual space of $\mathbb{R}^n$ is $\mathbb{R}^n$ itself.

(b) Let X be a normed space. Define the natural map Q from X into X^{**}. Prove that Q is an isometric isomorphism from X onto some subspace of X^{**}. Moreover, show that the subspace $Q(X)$ of X is closed if and only if X is a Banach space.

(c) Define three space property. Prove that separability is a three space property.

6. Answer ***any three*** parts : 5×3=15

(a) Consider the linear space $C_F[a, b]$ of all continuous F-valued functions on $[a, b]$. For $x, y \in C_F[a, b]$, define

$$\langle x, y \rangle = \int_a^b \overline{x(t)}\, y(t)\, dt .$$

Show that $(C_F[a, b], <,>)$ is an inner product space. Is $C_F[a, b]$ equipped with the norm

$$\|x\|_\infty = \sup_{t \in [a, b]} |x(t)|,\ x \in C_F[a, b]$$

a Hilbert space? Justify. 3+2=5

(b) Prove that a closed convex subset M of a Hilbert space X contains a unique vector of infimum norm.

(c) Define orthonormal set in an inner product space. Let $X = C[0, 2\pi]$ be the inner product space of all real valued continuous functions on $[0, 2\pi]$ with the inner product

$\langle x, y \rangle = \int_0^{2\pi} x(t)\, y(t)\, dt$. Show that $(x_n) = \cos nt$ and $(y_n) = \sin nt$, for $n = 0, 1, 2\ldots$ are orthogonal sequences of the space X. 1+4=5

(d) Let $x_n(t) = t^n$ for $t[-1, 1], n = 0, 1, 2, \ldots$ orthonormalize the first three terms of $\{x_n\}$, where the inner product is given by $\langle x, y \rangle = \int_{-1}^{1} x(t)\, y(t)\, dt$.

(e) Let X be a Hilbert space. If $f \in X^*$ is an arbitrary linear functional, then prove that there exists a unique vector $z \in X$ such that $f(x) = \langle x, z \rangle$, for every $x \in X$, where z depends on f and has the norm $\|z\| = \|f\|$.

7. Answer ***any three*** parts : 5×3=15

(a) Let $T : X \to X$ be an operator on a Hilbert space X. Prove that

(i) T is normal on X if and only if $\|T^* x\| = \|Tx\|$ for all $x \in X$;

(ii) the set of all normal operators on a Hilbert space X is a closed subset of $B(X)$, which contains the set of all self-adjoint operators and is closed under scalar multiplication. 2+3=5

(b) Let $T : X \to X$ be a bounded self-adjoint operator on a Hilbert space X. Prove that

(i) all the eigenvalues of T (if exists) are real;

(ii) eigenspaces corresponding to different eigenvalues of T are orthogonal. 2+3=5

(c) Define positive operator on a Hilbert space. Is the product of two positive operators on a Hilbert space again positive ? Justify with an example. 1+1+3=5

(d) Let $T : X \to X$ be a unitary operator on a Hilbert space X. Prove that

(i) T is isometric;

(ii) $\|T\| = 1$ provided $X \neq \{0\}$;

(iii) $T^{-1} (= T^*)$ is unitary;

(iv) T is normal. 2+1+1+1=5

(e) Consider the initial value problem $x'(t) = f(t, x(t))$ initial condition $x(t_0) = x_0$. Let f be continuous on a rectangle

$R = \{(t, x) : |t - t_0| \leq a, |x - x_0| \geq b\}$

and bounded on R say $|f(t, x)| \leq C$, for all $(t, x) \in R$. Using Banach fixed point theorem prove that the initial value problem has a unique fixed point in the space $X = C[a, b]$.