14 (MAT–2) 2046 (N/O)

2023

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Estd.-1964
Boko, Dist.- Kamrup

MATHEMATICS

(New and Old Course)

Paper : MAT–2046

(Number Theory)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any three*** parts : 5×3=15

   (a) Let order of a modulo n be k. Show that order of a^b modulo n is $\frac{k}{gcd(b, k)}$. Hence determine order of 3^{100} modulo 7. 4+1=5

   (b) *(i)* Let order of a modulo p be 3, where p is an odd prime. Show that $a+1$ must have order 6 modulo p.

(ii) Determine a primitive root of 10. 3+2=5

(c) If $gcd(m, n)=1$, $m>2, n>2$, then show that mn has no primitive root. Illustrate this result with an example. 3+2=5

(d) Construct a table of indices for the prime 13 with respect to a primitive root. Using this table solve the congruence $4x^9 \equiv 7 (mod\, 13)$. 2+3=5

2. Answer ***any three*** parts : 5×3=15

(a) What do you mean by the Legendre symbol $\left(\frac{a}{p}\right)$? Establish the following :

(i) $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right)=\left(\frac{ab}{p}\right)$

(ii) $\left(-\frac{1}{p}\right)=\begin{cases} 1 & \text{if } p \equiv 1 (mod\, 4) \\ -1 & \text{if } p \equiv 3 (mod\, 4) \end{cases}$

where p is an odd prime. 1+2+2=5

(b) (i) State Gauss lemma and use it to evaluate the Legendre symbol $\left(\frac{5}{13}\right)$.

(ii) Show that there are infinitely many primes of the form $4k+1$. 1+1+3=5

(c) State quadratic reciprocity law. Hence determine all primes p such that 3 is a quadratic residue of p and also determine all primes p such that 3 is a quadratic nonresidue of p. 1+2+2=5

(d) Determine whether the following congruences are solvable or not. If solvable then determine the solutions.

(i) $x^2+7x+10 \equiv 0 (mod\, 11)$

(ii) $3x^2+6x+5 \equiv 0 (mod\, 89)$ 2+3=5

3. Answer ***any three*** parts : 5×3=15

(a) Prove that *any* positive integer N can be expressed as $N = u_{k_1} + u_{k_2} + \ldots + u_{k_r}$ where $k_1 \geq 2$ and $k_{j+1} \geq k_j + 2$ for $j = 1, 2, \ldots, r-1$; and u_n is the n^{th} Fibonacci number.

(b) Prove that : 3+2=5

(i) $u_{n+1}^2 = u_n^2 + 3u_{n-1}^2 + 2\left(u_1^2 + u_2^2 + \ldots + u_{n-3}^2 + u_{n-2}^2\right)$

(ii) $u_{n+3} \equiv u_n \pmod 2$

where u_n is the n^{th} Fibonacci number.

(c) Using the Binet formula, prove that

$$u_{n+2}^2 - u_n^2 = u_{2n+2},$$

where u_n is the n^{th} Fibonacci number.

(d) Solve the linear diophantine equation using continued fraction :

$$172x + 20y = 1000$$

4. Answer ***any three*** parts : 5×3=15

(a) List all partitions of 5. Form the graphical representation of all the partitions and find the conjugate partition in each case. How many self-conjugate partitions of 5 can be found ?

(b) If $F(n)$ denotes the number of partitions of n with every part appearing at least twice and $G(n)$ denotes the number of partitions of n into parts large than 1 such that no two parts are consecutive integers, then show that $F(n) = G(n)$.

(c) Prove that the number of partitions of an integer n in which all parts are odd equals the number of partitions of n in which all parts are distinct.

(d) If $p(n)$ denotes the total number of partitions of n, then show that

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{j=1}^{\infty} \frac{1}{1-q^j}; \; |q| < 1$$

5. Answer ***any four*** parts : 5×4=20

(a) Prove that an irreducible polynomial over an algebraic number field has no repeated roots in $\mathbb{C}$.

(b) If g is a golden ratio and ρ_3 is a primitive cube root of unity, then show that $\mathbb{Q}(g, \rho_3) = \mathbb{Q}(g + \rho_3)$.

(c) Let $F = \mathbb{Q}(\rho_8)$, where ρ_8 is a primitive eighth root of unity. Then find all embeddings of F in $\mathbb{C}$ and prove the following : 3+1+1=5

(i) $\rho_8 + \rho_8^3 + \rho_8^5 + \rho_8^7 = 0$

(ii) $\rho_8^2 + \rho_8^6 = 0$

(d) Let $F = \mathbb{Q}(\rho_8)$, where ρ_8 is a primitive eighth root of unity. If $\beta = \sqrt{-1}$, then find the field polynomial $f(x)$ of β over F and show that

$$f(x) = m^2_{\beta, \mathbb{Q}}(x)$$

where $m_{\beta, \mathbb{Q}}(x)$ is the minimal polynomial of β over $\mathbb{Q}$.

3+2=5

(e) Let p be an odd prime and $F = \mathbb{Q}(\sqrt{p})$ such that $p \equiv \pm 3 \pmod 8$. Show that there is no $\alpha \in F$ such that $N_F(\alpha) = 2$.
