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14 (MAT–4) 4026 (N/O)

2023

MATHEMATICS

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Estd.-1964
Boko, Dist Kamrup

Paper : MAT–4026

(New/Old Course)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

(Advanced Functional Analysis)

1. Answer ***any four*** parts : 5×4=20

(a) Suppose that S is a subbasis for the topology of a toplogical space X, that $(x_\alpha)_{\alpha \in I}$ is a net in X, and that $x \in X$.

Prove that $x_\alpha \to x$ if and only if the following is true :
For every member U of S that contains x, there is an $\alpha_U \in I$ such that $x_\alpha \in U$ whenever $\alpha_U \leq \alpha$.

Contd.

(b) Let $\{X^{(\alpha)} : \alpha \in I\}$ be a family of topological spaces and let X be their topological product. Suppose that $(x_\beta)_{\beta \in J}$ is a net in X and $x \in X$. Prove that $x_\beta \to x$ if and only if $x_\beta^{(\alpha)} \to x^{(\alpha)}$, for each $\alpha \in I$.

(c) Prove that a subset S of a toplogical space is closed if and only if S contains every limit of every net whose terms lie in S.

(d) Suppose that X, Y and Z are topological spaces and that $(x,y) \mapsto x.y$ is a continuous operation from $X \times Y$ into Z. Suppose for then that (x_α) and (y_α) are nets in X and Y respectively having the same index set and that x and y are elements of X and Y respectively such that $x_\alpha \to x$ and $y_\alpha \to y$. Prove that $x_\alpha . y_\alpha \to x.y$.

(e) Let Y be a closed subspace of a locally convex space X. Suppose that $x \in X \setminus Y$. Show that there exists an $x^* \in X^*$ such that $x^* x = 1$ and $Y \subseteq ker(x^*)$.

2. Answer ***any two*** parts : 5×2=10

(a) Prove that every one-to-one bounded linear operator from an F-space onto an F-space has a bounded inverse.

(b) Let T be a linear operator from an F-space X into an F-space Y. Suppose that whenever a sequence (x_n) in X converges to some $x \in X$ and (Tx_n) converges to some $y \in Y$, it follows that $y = Tx$. Prove that T is continuous.

(c) Let (T_n) be a sequence of bounded linear operators from an F-space X into a Hausdorff topological vector space Y such that $\lim_n T_n x$ exists for each $x \in X$. Define $T : X \to Y$ by the formula $Tx = \lim_n T_n x$. Prove that T is a bounded linear operator from X into Y.

3. Answer ***any one*** part : 10

(a) Let M be a subspace of a locally convex space X. Prove that

(i) if $x_0 \in X$ and $x_0 \notin \overline{M}$, then there exists $\wedge \in X^*$ such that $\wedge x_0 = 1$, but $\wedge x = 0$ for every $x \in M$. If M is convex, balanced and closed, with $x_0 \notin M$, then there exists $\wedge \in X^*$ such that $|\wedge x| \leq 1$ for all $x \in M$, but $\wedge x_0 > 1$;

(ii) if f is a continuous linear functional on M, then there exists $\wedge \in X^*$ such that $\wedge = f$ on M.

5+5=10

(b) State and prove Krein-Milman theorem. 2+8=10

4. Answer ***any three*** parts : 5×3=15

(a) For a nonempty compact Hausdorff space K, let $C(K)$ be the collection of all complex valued continuous functions on K. Show that $C(K)$ is a Banach algebra with some suitable operations.

(b) Let A be a unital Banach algebra and $G(A)$ be the set of all invertible elements of A. Show that $G(A)$ is an open subset of A.

(c) Let A be a Banach algebra with unity e and $x \in A$. Prove that the spectrum $\sigma(x)$ is compact.

(d) Let A be a unital Banach algebra and $x \in A$. Let G be an open set in $\mathbb{C}$ such that $\sigma(x) \subset G$. Show that there exists $\delta > 0$ such that $\sigma(x+y) \subset G$ whenever $\|y\| < \delta$, $y \in A$.

5. Answer ***any two*** parts : 5×2=10

(a) Let Δ be the set of all complex homomorphisms on a commutative Banach algebra A. Show that every maximal ideal of A is the kernel of some $h \in \Delta$.

(b) Show that the maximal ideal space of a commutative Banach algebra A is a compact Hausdorff space.

(c) Show that every isomorphism between two semisimple commutative Banach algebras is a homeomorphism.

6. Answer ***any three*** parts : 5×3=15

(a) Define unitary element. If B is a commutative C^*-algebra, then prove that the set of all Hermitian elements of B is a closed linear subalgebra of B. 1+4=5

(b) Suppose that B is a commutative C^*-algebra. Prove that

(i) x is a Hermitian element of B if and only if Gelfand transform $\hat{x}$ of x is a real valued function.

(ii) if $u, v \in B$, $u \geq 0$, $v \geq 0$, then $u+v \geq 0$. 2+3=5

(c) Suppose that F is a positive functional on a Banach algebra B with an involution. Prove that

(i) $|F(x)|^2 \leq F(e)F(xx^*) \leq F(e)^2 \rho(xx^*)$

(ii) $|F(x)| \leq F(e)\rho(x)$, for every normal element $x \in B$. 3+2=5

(d) What do you mean by pure state in C^*-algebra? Give an example with justification. 1+4=5

(Dynamical System and Fractal Geometry)

(Symbols have their usual meaning)

1. Answer ***any three*** parts : 5×3=15

(a) Consider the initial value problem

$$\frac{dy}{dx} = y^{\frac{1}{3}}$$

$$y(0) = 0$$

Examine the existence and uniqueness of solution of the initial value problem in the rectangle $R : |x| \leq a, |y| \leq b$, a, b are non-negative real numbers.

(b) (i) Explain the concept of 'orbit of a map' with the help of 'square root' map.

(ii) Show the 'orbit analysis' for the map $f(x) = 5$. 2+3=5

(c) (i) Does the map $f(x) = x + 3$ has any fixed point? Justify.

(ii) What are the fixed points of the map $f(x) = x^2 - 2$?

(iii) Show that for the map $f(x) = x^2 - 1$, the points 0 and -1 lie on a periodic orbit of period 2. 1+2+2=5

(d) Use linear stability analysis method to find the nature of stability of the equilibrium points of

(i) $\dot{x} = \sin x$

(ii) $\dot{x} = x^2$ 3+2=5

(e) Explain what is meant by bifurcation with the help of the map $y = x^2 + c$, where c is a real parameter.

2. Answer ***any three*** parts : 5×3=15

(a) When is a bifurcation said to be a saddle-node bifurcation ? Explain it in case of one dimensional continuous dynamical system

$$\dot{x} = \alpha + x^2$$

where α is a real parameter.

(b) When is a bifurcation said to be a transcritical bifurcation ? Show that in case of the logistic map $f(x) = \mu x(1-x)$, a transcritical bifurcation takes place at the parameter value $\mu = 1$.

(c) What is meant by a Feigenbaum universal constant ? Explain how it helps us to find the approximate value of the third period-doubling bifurcation point once the first two period-doubling bifurcation points are known.

(d) Explain how one can find the period-two points of the logistic map analytically.

(e) Write a short note on the 'bifurcation diagram'. Give a rough sketch of the bifurcation diagram in the case of the logistic map.

3. Answer **either** *(a)* **or** *(b)* : 10

(a) Explain briefly what is meant by a limit cycle. Show that the non-linear system

$$\dot{x} = y + \frac{x}{\sqrt{x^2+y^2}}\left\{1-\left(x^2+y^2\right)\right\}$$

$$\dot{y} = -x + \frac{y}{\sqrt{x^2+y^2}}\left\{1-\left(x^2+y^2\right)\right\}$$

admits a limit cycle. Whether this limit cycle is stable ? Justify. 2+7+1=10

(b) What is meant by a Hopf bifurcation? Explain briefly. What is the nature of the eigenvalues in case of such bifurcation? Show that for the system

$$\frac{dx}{dt} = a - x - bx + x^2 y$$

$$\frac{dy}{dt} = bx - x^2 y$$

a Hopf bifurcation will occur if $a^2 + 1 = b$. 2+1+7=10

4. Write short notes on : ***(any two)*** 5×2=10

(a) Poincare section and Poincare map

(b) Chaos in the logistic map

(c) Strange attractors

5. Answer ***any three*** parts : 5×3=15

(a) Let $F \subseteq R^n$. Show that $H^s(F)$ is an outer measure, where

$H^s(F) = \lim_{\delta \to 0} \left\{ \sum |u_i|^s : \{u_i\} \right.$ is a δ-cover of $\left. F \right\}$.

(b) If $F \subseteq R^n$ and $f : F \to R^m$ be a mapping such that

$|f(x) - f(y)| \leq c|x - y|^\alpha (x, y \in F)$,

for constants $c > 0$, $\alpha > 0$. Then prove that for each s, $H^{s/\alpha}(f(F)) \leq c^{s/\alpha} H^s(F)$.

(c) Prove that a set $F \subseteq R^n$ with $\dim_H F < 1$ is totally disconnected.

(d) Write the construction of Sierpinski triangle. If S is the Sierpinski triangle and $s = \frac{\log 3}{\log 2}$, then prove that $\dim_H S = s$.

6. Answer ***any three*** parts : 5×3=15

(a) Let F be a subset of R^n whose box counting dimension is

$$dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

where $N_\delta(F)$ is the smallest number of sets of diameter δ that covers F.

Prove that $dim_B F = \lim_{\delta \to 0} \frac{\log N'_\delta(F)}{-\log \delta}$

where $N'_\delta(F)$ is the smallest number of closed balls of radius δ that covers F.

(b) If $F \subseteq R^n$, then prove that

(i) $\underline{dim_B} F = n - lim\ sup_{\delta \to 0} \frac{log\, Vol^n(F_\delta)}{log\, \delta}$

(ii) $\overline{dim_B} F = n - lim\ inf_{\delta \to 0} \frac{log\, Vol^n(F_\delta)}{log\, \delta}$

F_δ is a $\delta - nbhd$ of F.

(c) Let F be the middle-third Cantor set. Then prove that $\overline{dim_B} F = \underline{dim_B} F = \frac{log\, 2}{log\, 3}$.

(d) Prove that the box-counting dimension of $F = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, ...\right\}$ is $\frac{1}{2}$.
