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14 (MAT–4) 4046 (N/O)

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MATHEMATICS

Paper : MAT–4046

(New/Old Course)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

(Algebraic Topology)

1. Answer ***any four*** parts : 5×4=20

   (a) Let X be a topological space, u be a loop in X based at $x_0 \in X$ and c be the null loop in X at $x_0 \in X$. Prove that $u * u^{-1} \approx c\,rel\,\{0,1\}$.

   (b) What do you mean by a contractible space ? Show that every contractible space is path connected. 1+4=5

Contd.

(c) Let X be a topological space. If A is a deformation retract of X, then for any $a \in A$, prove that

$$\pi(A,a) \cong \pi(X,a).$$

(d) For any r-dimensional simplicial complex K, prove that

$$m(Sd\,K) \leq \left(\frac{r}{r+1}\right) m(K)$$

and so

$$m\left(Sd^n K\right) \leq \left(\frac{r}{r+1}\right)^n m(K).$$

(e) Prove that the diameter of an n-simplex $\sigma = (v_0, v_1, \ldots, v_n)$ is

$$\delta(\sigma) = max\left\{\left\| v_i - v_j \right\| : 0 \leq i, j \leq n\right\}.$$

2. Answer ***any four*** parts : 5×4=20

(a) Prove that $\Pi\left(S^1, \{1\}\right) \cong \mathbb{Z}$.

(b) Define singular $(n-1)$ simplex $\partial_i \phi$. Prove that

$\partial_n . \partial_{n+1} = 0$ for all $n \geq 0$. 1+4=5

(c) If X is a one-point space, then prove that $H_n(X) = 0$ for all $n \geq 0$.

(d) Prove that Hurewicz map

$$\Phi : \Pi_1(X, X_0) \to H_1(X)$$

is a homomorphism.

(e) Let X be any space and for $i = 0, 1$; let $\lambda^X : X \to X \times I$ be defined by $x \to (x, i)$. If $H_n\left(\lambda_0^X\right) = H_n\left(\lambda_1^X\right)$: $H_n(X) \to H_n(X \times I)$, then prove that $H_n(f) = H_n(g)$, whenever f and $g : X \to Y$ are homotopic.

3. Answer ***any two*** parts : 10×2=20

(a) What do you mean by lifting map ? Let X be a compact convex subset of $\mathbb{R}^n$ for some n. Let $f : (X, x_0) \to \left(S^1, 1\right)$ be continuous and $t_0 \in \mathbb{R}$ such that $\exp(t_0) = f(x_0)$, then prove that there exists a unique continuous map $f' : (X, x_0) \to (\mathbb{R}, t_0)$ such that $f'(x_0) = t_0$ and $(\exp) f' = f$. 2+8=10

(b) Let G be a path connected topological group and H be a discrete normal subgroup of G. Let $p : G \to \frac{G}{H}$ be the natural homomorphsim. Prove that (G, p) is a covering space of $\frac{G}{H}$.

(c) (i) Prove that (S^n, p) is a covering space of $\mathbb{R}p^n$, where p is the map identifying antipodal points of S^n for $n > 1$. 3

(ii) Show that $\pi_1(\mathbb{R}p^2, y) \cong \mathbb{Z}_2$. 3

(iii) What do you mean by topological manifolds? Verify that

$S^1 = \{(x, y) : x^2 + y^2 = 1\}$ is a manifold of dimension one and

$T^2 = \left\{(x, y\, z) : \left(\sqrt{x^2 + y^2} - 1\right)^2 + z^2 = \frac{1}{4}\right\}$ is a two dimensional surface of $\mathbb{R}^3$. 4

4. Answer ***any four*** parts : 5×4=20

(a) What is the importance of Mayer-Vietoris sequence? State the theorem of Mayer-Vietoris sequence. 3+2=5

(b) Define a vector field. State and prove the hairy ball theorem. 1+4=5

(c) State and prove Brouwer's fixed point theorem. 1+4=5

(d) Prove that S^n is not contractible for $n \geq 0$.

(e) Calculate the p^{th} homology of S^n.

1. *(a)* Calculate the non-vanishing Christoffel symbols of the second kind and obtain the differential equations of the geodesic for the metric

$$ds^2 = -dx^2 - dy^2 - dz^2 + f(x,y,z)\,dt^2 .$$

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Or

Derive the differential equations of a geodesic in the form

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0$$

(b) Define parallel displacement of a vector. Show that a geodesic is an autoparallel curve. 6

Or

Show that a vector of constant magnitude which suffers parallel displacement along a geodesic is inclined at a constant angle to the geodesic.

2. Answer ***any two*** of the following : 7½×2=15

(a) Show that the Riemann-Christoffel curvature tensor R^α_{ijk} can be contracted only in two ways—one of these leads to a symmetric tensor and the other to a zero tensor.

(b) Prove that

(i) $R^\alpha_{ijk} = -R^\alpha_{ikj}$

(ii) $R^\alpha_{ijk} + R^\alpha_{jki} + R^\alpha_{kij} = 0$

(iii) $R_{hijk} = R_{jkhi}$

(iv) $R_{hijk} + R_{hjki} + R_{hkij} = 0$

where the symbols have their usual meanings.

(c) Show that the Einstein tensor is divergence free.

3. Explain the clock paradox and resolve it with the help of the general theory of relativity. 10

Or

Derive the expression for the energy-momentum tensor T^{ij} for a perfect fluid distribution.

4. Answer ***any two*** of the following : 10×2=20

(a) Obtain the Schwarzschild exterior solution in the gravitational field of an isolated point particle.

(b) Derive the relativistic differential equation of a planetary orbit in the form

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2} + 3mu^2$$

with usual meanings of the symbols.

(c) Discuss the bending of light rays in a gravitational field.

5. (a) Stating the assumptions for static cosmological models, derive the de Sitter line-element for a static, isotropic and homogeneous universe. 10

Or

Discuss the density and pressure of matter in the Einstein universe.

(b) Discuss the geometry of the Friedmann-Lemaître-Robertson-Walker (FLRW) universe. 10

Or

Discuss the cosmological redshift in the Friedmann-Lemaître-Robertson-Walker universe.
